The non-commutative topology of dirty superconductors: the case of the Spin Hall effect

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Joint work with:
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1. BdG Hamiltonians and PH-symmetry
   - Tight-binding BdG Hamiltonians
   - Translational invariant pair potentials
   - Spectral consequences of PH-symmetry

2. Spin Hall conductance as a topological index
   - $SU(2)$ and TR invariant BdG Hamiltonians
   - Observable algebra of homogeneous operators
   - Real structure associated to PH-symmetry
   - Spin derivation and spin Hall current
   - Kubo formula for the spin Hall conductance

3. Spin Hall conductance for the $d + id$ model
   - Spectral properties of the $d + id$ model
   - The Bloch bundle for the $d + id$ model
   - Phase diagram for the spin Hall conductance
Overview on physics

- **Superconductivity**: zero electrical resistance and expulsion of magnetic fields for certain materials when \( T < T_{\text{crit}} \). Observed in 1911 (Onnes);

- **BCS-theory**: microscopic theory proposed in 1957 (Bardeen, Cooper & Schrieffer). Key notion: **Cooper pairs** (particle-hole), i.e. pairs of electrons interacting through the exchange of phonons. Nobel Prize in 1972;

- **BdG Hamiltonian**: derivation of the BCS wavefunction from a canonical transformation (Bogoliubov transformations) of the electronic Hamiltonian (Bogoliubov & de Gennes 1958);

- **Topological insulators**: BdG Hamiltonians provide models for **D-type** and **C-type** topological insulators: AZ classification (Altland & Zirnbaur, 1997);

- **Physical-topological effects**: spin quantum Hall effect (C-type) and thermal quantum Hall effect (D-type).
The spin quantum Hall conductance

\[ B_3(y) \equiv \text{Zeeman field}, \quad j^z_x := -\sigma^s_{xy} \left( \frac{dB_3}{dy} \right) \]
Tight-binding model

\[ h = \sum_{n,n' \in \Gamma} \sum_{l,l' = 1}^{r} h(n,n')_{l,l'} |n,l\rangle \langle n',l'| = \sum_{n,n' \in \Gamma} |n\rangle h(n,n') \langle n'| \]

- the “one-particle” Hilbert space is \( \mathcal{H} := \ell^2(\Gamma) \otimes \mathbb{C}^r \);
- \( \Gamma \) is some lattice. Mainly interested on \( \Gamma \simeq \mathbb{Z}^2 \);
- \( r \in \mathbb{N} \) internal degrees of freedom (spin + isospin);
- \( h(n,n') \) is a \( r \times r \) complex matrix for all \( n,n' \in \Gamma \);

Extra assumptions:

- self-adjointness \( h = h^\dagger \): this implies \( h(n,n') = h(n',n)^\dagger \);
- finite range (of size \( R \)): \( h(n,n') = 0 \) if \( |n - n'| > R \);
- space homogeneity: \( h \) can depend on “disorder” \( \omega \in \Omega \) in a “covariant” way

\[ h(n,n'; \tau_m(\omega)) = h(n + m, n' + m; \omega), \quad \forall \ m \in \Gamma. \]
Fock space representation and pairing potential

The second quantization functor is defined by

$$|n, l\rangle\langle n', l'| \mapsto c^\dagger_{n, l} c_{n', l'}.$$ 

Here $c^\dagger_{n, l}$ and $c_{n, l}$ are the creation and annihilation operators of a “spin state” $l$ on the site $n$ acting on the Fock space

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{N=0}^{\infty} \text{Antisym}(\mathcal{H} \otimes \ldots \otimes \mathcal{H})_{N-\text{times}}.$$ 

Under this map one has

$$h \mapsto h := \sum_{n, n' \in \Gamma} c^\dagger_n h(n, n') c_{n'} \equiv c^\dagger h c$$

with $c^\dagger_n := (c^\dagger_{n, 1}, \ldots, c^\dagger_{n, r})$ and $c^\dagger := \{c^\dagger_n\}_{n \in \Gamma}$.

$h$ preserves the particle number $N := \sum_n c^\dagger_n c_n \equiv c^\dagger c.$
The (fermionic) anticommutation relations

\[ \{ c^\dagger_{n,l}; c_{n',l'} \} = \delta_{n,n'} \delta_{l,l'}, \quad \{ c^\dagger_{n,l}; c^\dagger_{n',l'} \} = 0 = \{ c_{n,l}; c_{n',l'} \} \]

provide

\[ \sum_{n,n' \in \Gamma} c^\dagger_n h(n,n') c_{n'} = - \sum_{n,n' \in \Gamma} c_n h(n',n)^t c^\dagger_{n'} + C \mathbf{1}_\mathcal{F} \]

where \( C := \text{Tr}_{\mathcal{H}}[h] = \sum_{n \in \Gamma} \text{Tr}_{\mathcal{C}r}[h(n,n)] \).

The self-adjoint condition \( h(n',n)^t = \overline{h(n,n')} \) allows to write

\[ h = \frac{1}{2} c^\dagger h c - \frac{1}{2} c \overline{h} c^\dagger \]

Neglect (possibly infinite) \( \frac{1}{2} C \) means to do a renormalization.
BdG Hamiltonians and pair potentials

\[ H := \frac{1}{2} c^\dagger h c - \frac{1}{2} c^\dagger h c^\dagger + \frac{1}{2} c^\dagger \Delta c^\dagger - \frac{1}{2} c^\dagger \Delta c = h \]

pairing term

More precisely

\[ c^\dagger \Delta c^\dagger \equiv \sum_{n, n' \in \Gamma} c_n^\dagger \Delta(n, n') c_{n'}^\dagger \]

where \( \Delta(n, n') \) is a \( r \times r \) complex matrix for all \( n, n' \in \Gamma \).

Assumptions:

- **self-adjointness** \( H = H^\dagger \): this implies \( \Delta(n, n') = -\Delta(n', n)^\dagger \);
- **finite range** (of size \( R \)): \( \Delta(n, n') = 0 \) if \( |n - n'| > R \);
- **space homogeneity**: \( \Delta \) can depend on "disorder" \( \omega \in \Omega \) in a "covariant" way

\[ \Delta(n, n'; \tau_m(\omega)) = \Delta(n + m, n' + m; \omega), \quad \forall \quad m \in \Gamma. \]
Bogoliubov representation and PH-symmetry

Since $[H; N] \neq 0$ the BdG Hamiltonian $H$ does not preserves the particle number. However $H$ is quadratic and can be written in the Bogoliubov representation

$$H = \frac{1}{2} \left( c^\dagger c \right) \left( \begin{array}{cc} h & \Delta \\ -\overline{\Delta} & -h \end{array} \right) \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \equiv H.$$ 

The first quantized BdG Hamiltonian $H$ acts on the particle-hole Hilbert space $\mathcal{H}_{ph} := \mathcal{H} \otimes \mathbb{C}^2_{ph}$ (particle-hole fiber).

The PH-symmetry is

$$K H K^\dagger = -\overline{H}, \quad K := \left( \begin{array}{cc} 0 & 1 \mathcal{H} \\ 1 \mathcal{H} & 0 \end{array} \right).$$

Since $K^2 = 1_{\mathcal{H}_{ph}}$, then $H$ is a D-type topological insulator.
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Without disorder $h$ and $\Delta$ are chosen translational invariant.

Set $\Gamma = \mathbb{Z}^2$ and consider on $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^r$ the shift operator

$$(S_j \psi)(n) := \psi(n - e_j) \quad j = 1, 2.$$ 

along the canonical vector $e_j \in \mathbb{R}^2$.

The dynamical term $h$ (without magnetic field) is given by

$$h(\mu) := S_1 + S_1^\dagger + S_2 + S_2^\dagger - \mu 1_{\mathcal{H}}$$

(discrete laplacian)

where $\mu$ is the chemical potential. With magnetic field one can consider the Harper operator.

The pair potential $\Delta$ dictates the creation of Cooper pairs:

$$\Delta = \text{Polyn} [S_1, S_2]$$

such that $\Delta^\dagger = -\Delta$ (D-type).
d+i\,d pair potential (spin quantum Hall effect)

\[ \Delta_{d_{xy}} := (S_1 - S_1^\dagger)(S_2 - S_2^\dagger) \otimes i \sigma^2 \]  
(singlet \(d_{xy}\)-wave)

\[ \Delta_{d_{x^2-y^2}} := (S_1 + S_1^\dagger - S_2 - S_2^\dagger) \otimes i \sigma^2 \]  
(singlet \(d_{x^2-y^2}\)-wave)

\[ \Delta_{d_{\pm id}} := \Delta_{d_{x^2-y^2}} \pm i\alpha \Delta_{d_{xy}} \]  
(singlet \(d_{\pm i d}\)-wave)

where \(\sigma^j\) with \(j = 1, 2, 3\) are the Pauli matrices for spin \(s = \frac{1}{2}\) and \(\alpha \in \mathbb{R}\). \(d\)-wave potentials are even under the change \(S_j \leftrightarrow S_j^\dagger\).
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As a consequence of $K H K^\dagger = -H$ one has:

**Spectrum symmetry**

\[
\left( H + \bar{z} \mathbb{1}_{\mathcal{H}_{ph}} \right)^{-1} = -K^\dagger \left( H - z \mathbb{1}_{\mathcal{H}_{ph}} \right)^{-1} K
\]

implies

\[E \in \text{Spec}(H) \quad \iff \quad -E \in \text{Spec}(H).\]

**Symmetry of Fermi projection**

Let $f_\pm : \mathbb{R} \to \mathbb{R}$ such that $f_\pm(-x) = \pm f_\pm(x)$, then

\[K f_\pm(H) K^\dagger = \pm f_\pm(H).\]

Let $f_\beta(x) : = (1 + e^{-\beta x})^{-1}$ be the Fermi-Dirac function, then

\[K f_\beta(H) K^\dagger = \mathbb{1}_{\mathcal{H}_{ph}} - f_\beta(H).\]
Density of states and pseudo-gap

Let $\mathcal{N}$ be the integrated density of states (IDOS) for $H$ with normalization condition $\mathcal{N}(0) = 0$, i.e. for $E \geq 0$

$$\mathcal{N}_H(E) = T\left(\chi_{[0,E]}(H)\right), \quad \mathcal{N}_H(-E) = - T\left(\chi_{[-E,0]}(H)\right)$$

with $T$ the trace per unit volume (usual normalization $\mathcal{N}(-\infty) = 0$).

PH-symmetry $\implies \mathcal{N}_H(-E) = - \mathcal{N}_H(E)$

$\implies \mathcal{N}_H(E) = \frac{1}{2} \mathcal{N}_H^2(E^2)$.

If $\mathcal{N}_H$ and $\mathcal{N}_H^2$ are absolutely continuous with DOS $\rho$ and $\rho^{(2)}$ respectively, then one deduces

$$\rho(E) = |E| \rho^{(2)}(E^2).$$
Generic behavior for the DOS of BdG Hamiltonians

Schematic representation of the DOS $\rho^{(2)}$ and corresponding $\rho$ in the generic two situations: with true gap and pseudo-gap.
Non-generic behavior for the DOS of BdG Hamiltonians

Schematic representation of the DOS \( \rho^{(2)} \) and corresponding \( \rho \) for various non-generic cases.

In \( D = 2 \) the Van Hove singularities of \( \rho^{(2)} \) are at most of order \( E^{-(1-\varepsilon)} \) with \( \varepsilon > 0 \).
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Let $\sigma^1, \sigma^2, \sigma^3$ be an irreducible representation (hermitian and traceless) of $su(2)$ on $\mathbb{C}^r$ (spin $s = \frac{r-1}{2}$). $\sigma^1, \sigma^3 =$ real; $\sigma^2 =$ purely imaginary. The BdG second quantized is

$$\sigma^j := \frac{1}{2} (c^\dagger c) \begin{pmatrix} 1_{\ell^2} \otimes \sigma^j & 0 \\ 0 & -1_{\ell^2} \otimes \overline{\sigma^j} \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}.$$ 

Let $H$ be a second quantized BdG Hamiltonian associated to the first quantized operator $H$, then

$$[H; \sigma^j] = 0 \iff \left[ H; \begin{pmatrix} 1_{\ell^2} \otimes \sigma^j & 0 \\ 0 & -1_{\ell^2} \otimes \overline{\sigma^j} \end{pmatrix} \right] = 0.$$

- **U(1) - invariance**: $[H; \sigma^3] = 0$;
- **SU(2) - invariance**: $[H; \sigma^j] = 0$ for all $j = 1, 2, 3$. 
**THEOREM**

- Let $H$ be a $U(1)$-invariant (first quantized) BdG Hamiltonian, then

\[
H = \begin{pmatrix}
  h_1 & \cdots & h_r \\
  \cdots & \ddots & \cdots \\
  -\Delta_1 & \cdots & -h_r \\
\end{pmatrix}
\]

with $h_j^\dagger = h_j$ and $\Delta_j^\dagger = -\overline{\Delta_{r-j+1}}$ bounded on $\ell^2(\Gamma)$.

- Let $H$ be $SU(2)$-invariant, then

\[
h_1 = \ldots = h_r =: h_{\text{red}}, \quad \Delta_j = (-1)^{j+1} \Delta_{\text{red}},
\]

with $h_{\text{red}}^\dagger = h_{\text{red}}$ and $\Delta_{\text{red}}^\dagger = (-1)^r \overline{\Delta_{\text{red}}}$ bounded on $\ell^2(\Gamma)$. 
In the case of SU(2) - invariance

\[
H = \begin{pmatrix}
\begin{array}{c|c}
\hline
h_{\text{red}} \otimes \mathbf{1}_r & \Delta_{\text{red}} \otimes \mathbf{Y}_r \\
\hline
-\Delta_{\text{red}} \otimes \mathbf{Y}_r & -h_{\text{red}} \otimes \mathbf{1}_r
\end{array}
\end{pmatrix}
\]

with \( \mathbf{1}_r \) the \( r \times r \) identity matrix and

\[
\mathbf{Y}_r := \begin{pmatrix}
\begin{array}{ccc}
1 & & \\
& \ddots & -1 \\
(-1)^{r+1} & \cdots & 1
\end{array}
\end{pmatrix}.
\]

- \( \mathbf{Y}_r^\dagger = \mathbf{Y}_r^t = (-1)^{r+1} \mathbf{Y}_r \);
- \( \mathbf{Y}_r \sigma^3 = -\sigma^3 \mathbf{Y}_r \).
$H \equiv H_{\text{red}}^+ \oplus \ldots \oplus H_{\text{red}}^+ \oplus H_{\text{red}}^- \oplus \ldots \oplus H_{\text{red}}^-$

where $H_{\text{red}}^\pm$ are bounded operators on $\ell^2(\Gamma) \otimes C_{\text{ph}}^2$

\[
H_{\text{red}}^+ = \begin{pmatrix}
    h_{\text{red}} & \Delta_{\text{red}} \\
    (-1)^r & -h_{\text{red}}
\end{pmatrix}, \quad H_{\text{red}}^- = \begin{pmatrix}
    h_{\text{red}} & -\Delta_{\text{red}} \\
    (-1)^{r+1} & -h_{\text{red}}
\end{pmatrix}
\]

- $H_{\text{red}}^+$ and $H_{\text{red}}^-$ are unitarily equivalent

\[
J H_{\text{red}}^+ J^\dagger = H_{\text{red}}^-, \quad J := \begin{pmatrix}
    1_{\ell^2} & 0 \\
    0 & -1_{\ell^2}
\end{pmatrix}
\]

- $\Delta_{\text{red}}^\dagger = (-1)^r \Delta_{\text{red}}$ implies

\[
H_{\text{red}}^\pm = \begin{cases}
    \text{D-type} & \text{if } r \text{ odd} \\
    \text{C-type} & \text{if } r \text{ even}
\end{cases} \quad \rightarrow \quad (IHI^\dagger = -H, \quad I^2 = -1)
\]

$I = \text{symplectic matrix}$.
Time-reversal symmetry

The (first quantized) BdG Hamiltonian $H$ is **TR-invariant** if

\[ R h R^\dagger = \overline{h}, \quad R \Delta R^\dagger = \overline{\Delta} \quad R := 1_{\ell_2} \otimes e^{i\pi \sigma^2}. \]

Physical phenomena associated to BdG classes

<table>
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<tr>
<th>TRS</th>
<th>SU(2)</th>
<th>U(1)</th>
<th>AZ</th>
<th>PP</th>
<th>Effect</th>
<th>Invariant</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>D</td>
<td>$\Delta_{p\pm ip}$</td>
<td>TQHE</td>
<td>$\mathbb{Z}$</td>
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<td>±1</td>
<td>√</td>
<td>√</td>
<td>CI</td>
<td>$\Delta_{d_{xy}}, \Delta_{d_{x^2-y^2}}$</td>
<td>SQHE</td>
<td>0</td>
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<td>0</td>
<td>√</td>
<td>√</td>
<td>C</td>
<td>$\Delta_{d\pm id}$</td>
<td></td>
<td>$\mathbb{Z}$</td>
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</table>

Spatial dimension $D = 2$; spin half-integer ($r$ even);

(AZ) = Altland-Zirnbauer class of the (reduced) BdG operators;

(PP) = examples of pair potentials.
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   - Phase diagram for the spin Hall conductance
• Disorder: probability space \((\Omega, d\mathbb{P}, \tau)\) with: \(\Omega\) compact; \(\tau\) a \(\Gamma\)-action and \(d\mathbb{P}\ \tau\)-ergodic.

• Algebraic operations: on \(C_c(\Omega \times \Gamma) \otimes \text{Mat}(2r, \mathbb{C})\) one defines

\[
(A \ast B)(\omega, n) := \sum_{m \in \Gamma} A(\omega, m) \ B(\tau_m(\omega), m - n) \qquad n, m \in \Gamma, \ \omega \in \Omega.
\]

\[
A^*(\omega, n) := A(\tau_n(\omega), -n)^+.
\]

• Representations: for all \(\omega \in \Omega\) and \(\psi \in \mathcal{H}_{\text{ph}}\)

\[
(\pi_{\omega}(A)\psi)(n) := \sum_{m \in \Gamma} A(\tau_n(\omega), m - n) \ \psi(m).
\]

• \(C^*\)-norm: \(\|A\| := \sup_{\omega \in \Omega} \|\pi_{\omega}(A)\|_{\mathcal{B}(\mathcal{H}_{\text{ph}})}\).

• Observable algebra: \(\|\cdot\|\)-closure of \(C_c(\Omega \times \Gamma) \otimes \text{Mat}(2r, \mathbb{C})\)

\[
\mathcal{A} = C(\Omega) \times_{\tau} \Gamma \otimes \text{Mat}(r, \mathbb{C}) \otimes \text{Mat}(2, \mathbb{C}).
\]

\(\omega \mapsto \pi_{\omega}(A)\) is strongly continuous and \(\Gamma\)-covariant.
Differential calculus for $\mathcal{A}$

- **Spatial derivations (gradient):** $\nabla := (\nabla_1, \ldots, \nabla_d)$ defined by
  \[ (\nabla_j \mathcal{A})(\omega, n) := i (n \cdot e_j) \mathcal{A}(\omega, n) \]
  with $e_1, \ldots, e_d$ the generators of $\Gamma$. Let $X_j$ be the unbounded operator
  \[ (X_j \psi)(n) := (n \cdot e_j) \psi(n) \]
  for $\psi \in \mathcal{H} = \ell^2(\Gamma) \otimes \mathbb{C}^r$, then, on $\mathcal{H}_{ph} = \mathcal{H} \otimes \mathbb{C}^2_{ph}$
  \[ \pi_\omega (\nabla_j \mathcal{A}) = i \left[ \pi_\omega(\mathcal{A}); X_j \otimes 1_{ph} \right] . \]

- **Trace per unit volume (integration):** it is the trace state on $\mathcal{A}$ defined by
  \[ \mathcal{T}(\mathcal{A}) := \int_\Omega d\mu(\omega) \text{Tr}_{\ell^2} [\mathcal{A}(\omega, 0)] . \]
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Real structure

\( \kappa : \mathcal{A} \rightarrow \mathcal{A} \) is an anti-linear automorphism defined by

\[
\kappa(A) := K^\dagger \overline{A} K, \quad K := \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Observe that \( \kappa^2 = \text{Id} \) (involution).

- **Real algebra:** \( \mathcal{A}^\mathbb{R} := \{ A \in \mathcal{A} \mid \kappa(A) = A \} \). In the particle-hole grading

\[
A \in \mathcal{A}^\mathbb{R} \iff A = \begin{pmatrix}
a & b \\
b & \overline{a}
\end{pmatrix}, \quad a, b \in \left( C(\Omega) \rtimes \tau \Gamma \right) \otimes \text{Mat}(r, \mathbb{C}).
\]

- **Imaginary part:** the observable algebra \( \mathcal{A} \) splits as

\[
\mathcal{A} = \mathcal{A}^\mathbb{R} \oplus \mathcal{A}^\mathbb{I}, \quad \mathcal{A}^\mathbb{I} := i \mathcal{A}^\mathbb{R}.
\]

BdG Hamiltonians are elements of \( \mathcal{A}^\mathbb{I} \) (not an algebra!!)
• **Real K-theory:** $KR_0(\mathcal{A})$ is the set of homotopy classes of $P = P^2 = P^\dagger$ in $\mathcal{A} \otimes \text{Mat}(\infty, \mathbb{C})$ such that $\kappa(P) = P$.

• **Fermi projections:** for a BdG Hamiltonians one has

$$\kappa(P) = 1 - P, \quad \kappa(P) = 1 - P$$

which implies $\mathcal{F}(P) = \frac{1}{2}$ (half-dimensionality);

• **Grothendieck group structure:** in $K_0(\mathcal{A})$ one verifies

$$\kappa([1 - P], [\overline{P}]) \simeq ([\overline{P}], [1 - P]) \equiv "-" ([1 - P], [\overline{P}]).$$

• **Imaginary K-theory !?:** The set of $([Q_1], [Q_2]) \in K_0(\mathcal{A})$ s.t.

$$\kappa([Q_1], [Q_2]) = ([Q_2], [Q_1])$$

defines a subgroup $KI_0(\mathcal{A})$ in $K_0(\mathcal{A})$. The Fermi projection of a BdG Hamiltonian select an element in $KI_0(\mathcal{A})$. 
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\[ \mathcal{A} = \left( \mathbb{C}(\Omega) \rtimes \Gamma \right) \otimes \text{Mat}(r, \mathbb{C}) \otimes \text{Mat}(2, \mathbb{C}) . \]

- **U(1)** and **SU(2)**-invariant subalgebras:

\[ \mathcal{A}_{U(1)} := \left\{ A \in \mathcal{A} \mid [A; 1 \otimes \sigma^3 \otimes J] = 0 \right\} \]

\[ \mathcal{A}_{SU(2)} := \left\{ A \in \mathcal{A} \mid [A; 1 \otimes \sigma^i \otimes J] = [A; 1 \otimes \sigma^2 \otimes 1_{ph}] = 0, \quad i = 1, 3 \right\} \]

where \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). One has \( \mathcal{A}_{SU(2)} \subset \mathcal{A}_{U(1)} \subset \mathcal{A} \). Moreover

\[ \mathcal{A}_{\mathbb{R}, \mathbb{I}}^{U(1)} := \mathcal{A}_{U(1)} \cap \mathcal{A}_{\mathbb{R}, \mathbb{I}}, \quad \mathcal{A}_{\mathbb{R}, \mathbb{I}}^{SU(2)} := \mathcal{A}_{SU(2)} \cap \mathcal{A}_{\mathbb{R}, \mathbb{I}} . \]

- **Spin derivation**: on \( \mathcal{A}_{U(1)} \cap \mathcal{A}^1 \) one defines

\[ \nabla^S_j A := (\nabla^j A) (1 \otimes \sigma^3 \otimes J), \quad j = 1, 2, 3. \]

The \( \nabla^S_j \) are not self-adjoint derivations (Leibnitz rule) on \( \mathcal{A}^1 \).
• **Spin current:** if $A \in \mathcal{A}_{U(1)} \cap \mathcal{A}^1$ then

$$\pi_\omega (\nabla^s_j A) = i [\pi_\omega (A); X_j \otimes 1_{ph}] (1 \otimes \sigma^3 \otimes J).$$

If $H \in \mathcal{A}_{U(1)} \cap \mathcal{A}^1$ is a BdG Hamiltonian, then

$$\mathcal{J}^{\text{spin}}_j (\omega) := \pi_\omega (\nabla^s_j H) = i [\pi_\omega (H); X_j \otimes J] \left( 1 \otimes \sigma^3 \otimes 1_{ph} \right).$$

$\mathcal{J}^{\text{spin}} \propto \vec{v} \sigma^3$ is a current of spin for physicists.

• **Equilibrium condition:** let $A \in \mathcal{A}_{SU(2)}$ and $B \in \mathcal{A}_{SU(2)} \cap \mathcal{A}^1$:

$$\mathcal{T} (A \ast (\nabla^s_j B)) = 0.$$

If $H \in \mathcal{A}_{SU(2)} \cap \mathcal{A}^1$ is a BdG Hamiltonian and $\rho := f_\beta (H)$ is the Fermi-Dirac (equilibrium) state, then

$$\langle \mathcal{J}^{\text{spin}} \rangle_\rho := \mathcal{T} (\rho \ast (\nabla^s H)) = 0.$$
Outline

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Linear response theory

We need to perturb the BdG Hamiltonian $H \in \mathcal{A}_{SU(2)} \cap \mathcal{A}^1$ with a Zeeman field $\lambda \mathcal{P}$ with

$$\mathcal{P} := \chi^2 \otimes \sigma^3 \otimes J, \quad \lambda \propto \vec{B} \cdot \hat{z}.$$  

- The Liouville operators associated to $H$ and $\mathcal{P}$ are

$$\mathcal{L}_H(A) := i[A; H], \quad \mathcal{L}_\mathcal{P}(A) := \nabla^s_2 A.$$  

- The operator $\mathcal{L}_H + \lambda \mathcal{L}_\mathcal{P}$ defines a time evolution on $\mathcal{A}_{U(1)}$.

- We consider the $T = 0$ equilibrium state $\mathcal{P} = \chi_{(-\infty,0]}(H)$.

- The spin Hall conductance in the direction $1 \equiv x$ for a Zeeman field depending on the direction $2 \equiv y$ is given by

$$\sigma^s_{x,y} := \lim_{\delta \to 0} \mathcal{T}\left( \left( \nabla^s_2 \mathcal{P} \right) \left[ \delta + \mathcal{L}_H \right]^{-1} \left( \nabla^s_1 H \right) \right) \quad \text{(Kubo formula)}.$$
**Theorem (D. & Schulz-Baldes, 2013)**

Let \( H \in \mathcal{A}_{SU(2)} \cap \mathcal{A}^1 \) and \( P = \chi_{(-\infty,0]}(H) \). Assume that:

(a) \( \mathcal{T}(|\nabla^s P|^2) < +\infty \), \( (localization \ condition) \);

(b) The DOS of \( H \) has no atom in \( E = 0 \).

Then the Kubo-Chern formula holds:

\[
\sigma_{1,2}^s = \mathrm{i} \mathcal{T} \left( P \left[ \nabla_1^s P; \nabla_2^s P \right] \right) \\
= \frac{1}{8\pi} 2\pi \mathcal{T} \left( P \left[ \nabla_1 P; \nabla_2 P \right] \right) \in \frac{1}{8\pi} \mathbb{Z} \\
\text{Chern number of } P
\]

**Proof (sketch of).**

\( H \in \mathcal{A}_{SU(2)} \Rightarrow P , \nabla_j P \in \mathcal{A}_{SU(2)} \) then

\[
(\nabla_i^s P) * (\nabla_j^s P) = (\nabla_i P) * (\nabla_j P) * (1 \otimes \sigma^3 \otimes J)^2 = \frac{1}{4} (\nabla_i P) * (\nabla_j P) .
\]
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d+id wave model in Fourier variables

\[ H_{\mu,\alpha}(k_1, k_2) = 2 \sum_{i=1,2,3} f_i(k_1, k_2) \otimes \Sigma_i \]

with \((k_1, k_2) \in [0,1)^2 \sim T^2\) (Brillouin torus),

\[ f_1(k_1, k_2) := \cos(2\pi k_1) - \cos(2\pi k_2) \]
\[ f_2(k_1, k_2) := 2\alpha \sin(2\pi k_1) \sin(2\pi k_2) \]
\[ f_3(k_1, k_2) := \cos(2\pi k_1) + \cos(2\pi k_2) + \frac{\mu}{2} \]

and, introducing \(\gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\),

\[ \Sigma_1 := i \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \Sigma_2 := \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}, \quad \Sigma_3 := \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}. \]

Observe that \(\Sigma_i \Sigma_j = i\epsilon_{ijk} \Sigma_k + \delta_{ij} 1_4\) (Pauli matrices algebra).
Spectral analysis

- Dispersion relation: two energy bands $E_{\pm}$ given by

$$E_{\pm}(k_1, k_2) := \pm r(k_1, k_2), \quad r(k_1, k_2) := \left(\sum_{i=1}^{3} f_i(k_1, k_2)^2\right)^{\frac{1}{2}}$$

**Theorem**

The $d+i d$ wave BdG Hamiltonian $H_{\mu, \alpha}$ as a gap around zero if and only if $\mu \neq 0, \pm 4$ and for all $\alpha \in \mathbb{R} \setminus \{0\}$.

- Polar variables: together with $r$ one introduces two angles

$$\begin{cases} 
\varphi(k_1, k_2) := \arctan \left( \frac{f_2(k_1, k_2)}{f_1(k_1, k_2)} \right) & 0 \leq \varphi(k_1, k_2) < 2\pi \\
\theta(k_1, k_2) := \arccos \left( \frac{f_3(k_1, k_2)}{r(k_1, k_2)} \right) & 0 \leq \theta(k_1, k_2) \leq \pi .
\end{cases}$$
• Eigenvector: the gap condition $r(k_1, k_2) \neq 0$ allows to rewrite

$$H_{\mu, \alpha}(k) = r \begin{pmatrix}
\cos[\theta] & 0 & 0 & \sin[\theta] e^{-i\varphi} \\
0 & \cos[\theta] & -\sin[\theta] e^{-i\varphi} & 0 \\
0 & -\sin[\theta] e^{i\varphi} & -\cos[\theta] & 0 \\
\sin[\theta] e^{i\varphi} & 0 & 0 & -\cos[\theta]
\end{pmatrix}.$$ 

The eigenvectors for the negative energy band are

$$\Psi_1(k) := \begin{pmatrix}
\sin\left[\frac{1}{2}\theta\right] e^{-i\varphi} \\
0 \\
0 \\
-\cos\left[\frac{1}{2}\theta\right]
\end{pmatrix}, \quad \Psi_2(k) := \begin{pmatrix}
0 \\
\sin\left[\frac{1}{2}\theta\right] e^{-i\varphi} \\
\cos\left[\frac{1}{2}\theta\right] \\
0
\end{pmatrix}.$$ 

• Critical Points: the polar representation is singular in

$$\mathcal{N} := \left\{ k \in \mathbb{T}^2 \mid f_1(k) = 0 = f_2(k), \quad f_3(k) > 0 \right\} \quad \text{(north pole)}$$

$$\mathcal{S} := \left\{ k \in \mathbb{T}^2 \mid f_1(k) = 0 = f_2(k), \quad f_3(k) < 0 \right\} \quad \text{(south pole)}.$$
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• The vector bundle: hermitian and rank-two $\pi : \mathcal{E}_- \to \mathbb{T}^2$

$$\mathcal{E}_- := \bigsqcup_{k \in \mathbb{T}^2} \text{Ran}[P_-(k)] = \bigsqcup_{k \in \mathbb{T}^2} \left\{ v \in \mathbb{C}^4 \mid P_-(k)v = v \right\}$$

where $P_-(k)$ is the (globally defined) negative energy projection

$$P_-(k) := |\psi_1(k)\rangle\langle\psi_1(k)| + |\psi_2(k)\rangle\langle\psi_2(k)|$$

• Local trivialization: in $\mathbb{T}^2 \setminus \mathcal{S}$ is given by the maps $\psi_1(k)$ and $\psi_2(k)$. In $\mathbb{T}^2 \setminus \mathcal{N}$ we can use the “twisted” eigenvectors

$$\tilde{\psi}_1(k) := e^{i\varphi(k)} \psi_1(k), \quad \tilde{\psi}_2(k) := e^{i\varphi(k)} \psi_2(k).$$

• Transition functions: if both $\mathcal{N} \neq \emptyset$ and $\mathcal{S} \neq \emptyset$ then $\mathcal{E}_-$ has (non trivial) transition functions

$$g_{\mathcal{N}, \mathcal{S}}(k) := \begin{pmatrix} e^{i\varphi(k)} & 0 \\ 0 & e^{i\varphi(k)} \end{pmatrix}, \quad k \in \mathbb{T}^2 \setminus \{\mathcal{N}, \mathcal{S}\}.$$
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Chern invariant as a winding number

- Singular points for various regimes: for all $\alpha \neq 0$

$$\begin{cases} 
\mathcal{N} = \{k_*, k_{**}\}, \quad \mathcal{I} = \emptyset & \text{if } \mu > 4 \quad (I) \\
\mathcal{N} = \emptyset, \quad \mathcal{I} = \{k_*, k_{**}\} & \text{if } \mu < -4 \quad (II) \\
\mathcal{N} = \{k_*\}, \quad \mathcal{I} = \{k_{**}\} & \text{if } -4 < \mu < 4 \quad (III)
\end{cases}$$

with $k_* = (0, 0)$ and $k_{**} = (\frac{1}{2}, \frac{1}{2})$. The topology is non-trivial only in the case III.

- Chern-Euler class: we need the following facts:
  - $[c_1]$ is the top Chern class on $\mathbb{T}^2$;
  - the top Chern class equals the Euler class, i.e. $[c_1] = [e]$;
  - $[c_1](\mathcal{E}_-) = [c_1](\det(\mathcal{E}_-))$, the determinate line-bundle;
  - the Euler class for the line-bundle $\det(\mathcal{E}_-)$ is given by

$$[e] = d \left[ -\frac{1}{2\pi} d \left( \frac{1}{i} \log \det(g_{\mathcal{N},\mathcal{I}}) \right) \right] = d \left[ -\frac{1}{\pi} d\varphi \right]$$
The Chern-number: the Stokes’ theorem provides

\[ C_1(\mathcal{E}_-) := \int_{\mathbb{T}^2} [e](\mathcal{E}_-) = \frac{1}{\pi} \int_{\Gamma_N} d\varphi \]

with \( \Gamma_N \) a regular closed paths which wrap in counterclockwise direction the set \( N \). Then we can parametrized \( \Gamma_N \) by

\[ \{0, 2\pi\} \ni t \mapsto \xi_\epsilon(t) := \frac{\epsilon}{2\pi} (\cos(t), \sin(t)) , \quad \epsilon > 0 . \]

Since \( d\varphi = F_1(k) \, dk_1 + F_2(k) \, dk_2 \) one has

\[ C_1(\mathcal{E}_-) = \frac{1}{\pi} \int_0^{2\pi} F(\xi_\epsilon(t)) \cdot \dot{\xi}_\epsilon(t) \, dt . \]
Residue integral: one computes $C_1$ in the limit $\varepsilon \to 0$,

$$F(\xi_\varepsilon(t)) \cdot \dot{\xi}_\varepsilon(t) = \frac{2\alpha}{\cos(2t)^2 + 4\alpha^2 \sin(2t)^2} + O(\varepsilon)$$
Phase diagram for $C_1$ in the $d+i d$ model $H_{\mu, \alpha}$
Thank you for your attention