Geometric Aspects of Quantum Condensed Matter

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Lecture V

An Introduction to Bloch-Floquet Theory (part. I)

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An Introduction to Bloch-Floquet Theory

- Recommended Bibliography
- A Simple Prototypical Example
- $\mathbb{Z}^d$-symmetries
“Classics” for the Bloch-Floquet theory:


Algebraic approach to Bloch-Floquet theory:


Bloch-bundle:


1. An Introduction to Bloch-Floquet Theory
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Let $G = \{\text{Id}, g_1, \ldots, g_\ell\}$ be a finite Abelian group (FAG) of order $\ell$.

The fundamental representation theorem for FAG states that

$$G \simeq \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_N}$$

$$\left( \prod_{j=1}^{N} p_j = \ell \right)$$

where $\mathbb{Z}_{p_j} = \{[0], [1], \ldots, [p_j - 1]\}$ is the cyclic group of order $p_j \in \mathbb{N}$.

Let $I_j := \{0, 1, \ldots, p_j - 1\}$ and define the spectrum

$$\hat{G} := I_1 \times \ldots \times I_N.$$ 

For $k := (k_1, \ldots, k_N) \in \hat{G}$ let $g_k \in G$ defined as $g_k := ([k_1], \ldots, [k_N])$. We notice that $|\hat{G}| = |G| = \ell$.

Let $\mathcal{H}$ be a separable Hilbert space and $U : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of $G$. Assume that:

(a) $U$ is faithful, i.e. $U_g = \mathbb{1}$ if and only if $g = \text{Id}$;

(b) $U$ is algebraically compatible, i.e. the operators $\{U_g | g \in G\}$ are linearly independent over $\mathcal{H}$. 
The full representation is generated by $N$ unitary generators

$U_1 := U([1],[0],\ldots,[0])$, $U_2 := U([0],[1],\ldots,[0])$, \ldots $U_N := U([0],[0],\ldots,[1])$.

Moreover, we can use the spectrum $\hat{G}$ to label the unitary operators

$\hat{G} \ni k = (k_1, k_2, \ldots, k_N) \mapsto U_{g_k} = U_1^{k_1} U_2^{k_2} \ldots U_N^{k_N}$.

The order is not important since the generators $U_j$ commute.

The condition

$U_j^{p_j} = U([0],\ldots,[p_j],\ldots,[0]) = U([0],\ldots,[0],\ldots,[0]) = U_{\text{Id}} = 1$

implies that the eigenvalues of $U_j$ have the form $e^{i \frac{2\pi n}{p_j}}$.

**Problem (Spectral analysis of the representation $U$):**

- Compute eigenvalues and eigenspaces for each generator $U_j$;
- Build the simultaneous diagonalization of the full family $\{U_g | g \in G\}$;
- Let $\mathcal{A}(G)$ the commutative $C^*$-algebra generated by the family $\{U_g | g \in G\}$. Compute the Gel’fand spectrum of $\mathcal{A}(G)$. 
All the answers are contained in the following formula

\[ P_t := \frac{1}{|G|} \sum_{g \in G} \chi_t(g) \ U_g = \frac{1}{\ell} \sum_{k \in \hat{G}} \left( e^{i 2\pi \frac{t_1}{p_1}} \right)^{k_1} \ldots \left( e^{i 2\pi \frac{t_N}{p_N}} \right)^{k_N} U_{1}^{k_1} \ldots U_{N}^{k_N} =: \chi_t(g_k) \]

defined for all \( t = (t_1, \ldots, t_N) \in \hat{G}. \) The map \( \chi_t : G \to S^1 \) is called character.

**PROPOSITION (Bloch-projection)**

*For all \( t \in \hat{G} \) the operator \( P_t \) is an orthogonal projection.* Moreover:

(i) \( P_t \ P_t' = \delta_{t,t'} \ P_t \) for all \( t, t' \in \hat{G}; \)

(ii) \( \bigoplus_{t \in \hat{G}} P_t = P_{(0,\ldots,0)} = 1; \)

(iii) \( U_g \ P_t = \chi_t(g)^{-1} \ P_t \) for all \( t \in \hat{G} \) and \( g \in G. \)

The map \( \hat{G} \ni t \to P_t \in \text{Proj}(\mathcal{H}) \) is called **Bloch-projection**. This induces an orthogonal decomposition of the Hilbert space

\[ \mathcal{H} = \bigoplus_{t \in \hat{G}} \mathcal{H}_t =: \bigoplus_{t \in \hat{G}} \text{Im}(P_t) \]

If \( \psi = \bigoplus_t \psi_t \) is the decomposition of \( \psi \in \mathcal{H}, \) then \( U(g)\psi = \bigoplus_t \chi_t(g)^{-1} \psi_t. \)
Proof (sketch of).

The relation \( P_t = P_t^* \) follows observing that the adjoint change \( k_j \to p_j - k_j \), i.e. it amounts only to a relabeling of \( k \) in the definition of \( P_t \).

(i) implies that \( P_t \) is a projection and it can be proved from the equation

\[
P_t P_t' = \frac{1}{|G|^2} \sum_{g,g' \in G} \chi_t(g) \chi_{t'}(g') \ U_{g+g'} = \left( \frac{1}{|G|} \sum_{g \in G} \chi_{t'}(g) \right) P_t
\]

and the cyclotomic identity

\[
\sum_{k_j=0}^{p_j-1} \left( e^{i 2\pi \frac{t}{p_j}} \right)^{k_j} = p_j \delta_{r,0}, \quad \text{(why is it true?!)}
\]

(ii) can be proved using linearity and \( \sum_{t \in \hat{G}} \chi_t(g) = \ell \delta_{g,\text{Id}} \) which follows again from the cyclotomic identity.

(iii) \( U_j P_t = e^{-i 2\pi \frac{t_j}{p_j}} P_t \) for each generator \( j = 1, \ldots, N \).
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We want to generalize the Bloch decomposition to the case of finitely generated Abelian groups (FGAG).

- We recall that an Abelian group $G$ if FGAG if there exist finitely many element $\{g_1, \ldots, g_N\} \subset G$ such that every $g \in G$ can be written in the form
  $$g = n_1 g_1 + \ldots + n_N g_N, \quad n_1, \ldots, n_N \in \mathbb{Z}$$
In this case, we say that the set $\{g_1, \ldots, g_N\}$ is a generating set of $G$.

- The fundamental theorem of FGAG says that if $G$ is a FGAG then
  $$G \cong \mathbb{Z}^d \oplus \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_\ell}$$
  where the $d \geq 0$ is the dimension of the free part and the numbers $p_1, \ldots, p_\ell$ are powers of (not necessarily distinct) prime numbers which are uniquely determined by $G$. The group is finite if and only if $d = 0$.

- Since we know the Bloch decomposition for the torsion part (i.e. when $d = 0$) we will assume in the following that $G$ is purely free, namely
  $$G \equiv \mathbb{Z}^d, \quad d \in \mathbb{N}.$$
Let $\mathcal{H}$ be a separable Hilbert space and $U : \mathbb{Z}^d \to \mathcal{U}(\mathcal{H})$ a unitary representation, namely
- $U(n)$ is an unitary operator for each $n \in \mathbb{Z}^d$;
- $U(0) = 1$ and $U(n)U(m) = U(n + m)$

Moreover, we say that:
(a) $U$ is faithful, i.e. $U(n) = 1$ if and only if $n = 0$;
(b) $U$ is algebraically compatible, i.e. the collection operators $\{U(n) | n \in \mathbb{Z}^d\}$ is linearly independent over $\mathcal{H}$.

**Definition ($\mathbb{Z}^d$-algebra)**

Let $U : \mathbb{Z}^d \to \mathcal{U}(\mathcal{H})$ be a unitary representation faithful and algebraically compatible. The commutative $C^*$-algebra $\mathcal{Z}(d) \subset \mathcal{B}(\mathcal{H})$ generated by the set $\{U(n) | n \in \mathbb{Z}^d\}$ is a $\mathbb{Z}^d$-algebra.

**Remark:** Let $\mathcal{H}$ be a Hilbert space which carries a $\mathbb{Z}^d$-algebra $\mathcal{Z}(d)$. Topological insulators are (self-adjoint) operators $H$ defined on $\mathcal{H}$ which are invariant with respect $\mathcal{Z}(d)$. More precisely $H$ is a topological insulator if
$$[H, A] = 0 \ , \quad \forall \ A \in \mathcal{Z}(d) \ .$$
Example (Periodic Systems): Let $e_j := (0, \ldots, 1, \ldots, 0)$ with $j = 1, \ldots, d$ be the canonical basis of $\mathbb{R}^d$ and $\mathbb{Z}^d$.

- Continuous (twisted) translations:
  On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ one defines the unitary operators $T_1, \ldots, T_d$
  $$ (T_j \psi)(x) := g_j(x) \psi(x - e_j) \quad \psi \in L^2(\mathbb{R}^d) $$
  where the twist functions are normalized $|g_j| = 1$ and $\mathbb{Z}^d$-periodic $g_j(\cdot + n) = g_j(\cdot)$. The map $n \mapsto T(n)$ defined by
  $$ T(n_1, \ldots, n_d) := (T_1)^{n_1} \ldots (T_d)^{n_d} \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}^d $$
  defines a $\mathbb{Z}^d$-algebra on $L^2(\mathbb{R}^d)$.

- Discrete (twisted) translations:
  On the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ one defines the unitary operators $T_1, \ldots, T_d$
  $$ (T_j \psi)(n) := c_j \psi(n - e_j) \quad \psi \in \ell^2(\mathbb{Z}^d) $$
  where $c_j \in S^1$ are called twist constants. The map $n \mapsto T(n)$ defined by
  $$ T(n_1, \ldots, n_d) := (T_1)^{n_1} \ldots (T_d)^{n_d} \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}^d $$
  defines a $\mathbb{Z}^d$-algebra on $\ell^2(\mathbb{Z}^d)$. 
**Example (Mathieu-like System):** Let \( T := \mathbb{R}/(2\pi \mathbb{R}) \cong [-\pi, +\pi] \) be the 1-dimensional torus and consider the Hilbert space \( \mathcal{H} = L^2(T) \). A basis is provided by the Fourier system

\[ \xi_j(k) := \frac{1}{\sqrt{2\pi}} e^{ikj}, \quad j \in \mathbb{Z}. \]

Let us define two unitary operators \( u \) and \( v \) by

\[ (u\phi)(k) := e^{ik} \phi(k), \quad (v\phi)(k) := \phi(k - 2\pi\alpha) \quad \phi \in L^2(T). \]

One verifies that \( uv = e^{i2\pi\alpha} vu \). If the rational condition \( \alpha = N/M \) holds true then \( [u, v^M] = 0 \). In this case the map

\[ (n_1, n_2) \mapsto T(n_1, n_2) := (u)^{n_1} (v)^{Mn_2} \quad (n_1, n_2) \in \mathbb{Z}^2 \]

defines a \( \mathbb{Z}^2 \)-algebra on \( L^2(T) \). In this case the dimension of the group \( \mathbb{Z}^2 \) is bigger than the spatial dimension of the manifold \( T \).