Open dynamical systems for beginners: algebraic foundations

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Introduction

Open system theory is deeply connected with stochastic analysis foundation in both, commutative and non-commutative versions. Biologist von Bertalanffy pointed out in 1950 the importance of defining living matter as an open-dynamical-system. In parallel, physicists working on radiation theory introduced quantum dynamical semigroups to model the interaction between a system and its reservoir. And the open-system point of view invaded numerous fields like Finance, supported by stochastic differential equations, as well as Markov processes and many other probabilistic branches.

These lectures provide a panorama of the algebraic setting which allows to synthesize commutative and non-commutative open-system dynamics via semigroup theory. I do not suppose any previous knowledge of Quantum Mechanics and will not develop applications to Physics which have been the object of a number of research papers and books, some of them quoted in the references. Our goal is simply to explain the passage from the customary classical Markov Theory to the non commutative one. I have in mind that the reader has a basic knowledge of classical Stochastic Analysis. Markov Theory is especially well adapted to deal with classical open systems, which are described by stochastic differential equations. Stochastic differential equations, Markov processes, Markov semigroups are all connected with mathematical descriptions of open systems in classical Physics. So do quantum stochastic differential equations, quantum flows and quantum Markov semigroups, which are mathematical descriptions of open quantum systems. In both cases one looks for a memoryless approach to the dynamics of a composed system. However, although their similarities classical and quantum Markov semigroups have a deep difference: observables and dynamics have to suitably describe the Uncertainty Principle in the quantum case. This forces us to deal with non-commutative stochastic analysis and non-commutative geometry, both deeply founded in operator algebras.
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Chapter 1

From Newton to Langevin

We denote by $\Sigma \subseteq \mathbb{R}^3 \times \mathbb{R}^3$ the state space of a single particle mechanical system, that is, each element $x = (q, p) = (q_1, q_2, q_3, p_1, p_2, p_3) \in \Sigma$ corresponds to the pair of position and momentum of a particle, which is supposed to have mass $m$.

In Newtonian Mechanics, the dynamics is entirely characterised by the Hamilton operator which, in the homogeneous case, can be written as

$$H(x) = \frac{1}{2m} |p|^2 + V(q), \quad (x \in \Sigma), \quad (1.0.1)$$

where $|\cdot|$ denotes here the euclidian norm in $\mathbb{R}^3$. This allows to write the initial value problem which characterises the evolution of states as

$$\begin{cases} q'_i = \frac{\partial H}{\partial p_i}(x), \\ p'_i = -\frac{\partial H}{\partial q_i}(x), \quad 1 \leq i \leq 3, \\ x(0) = x_0. \end{cases} \quad (1.0.2)$$

If $I$ denotes the $3 \times 3$ identity matrix call

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (1.0.3)$$

Then (1.0.2) becomes

$$x' = J \nabla H(x), \quad x(0) = x_0, \quad (1.0.4)$$

where $\nabla$ is the customary notation for the gradient of a function.

1.1 The classical flow

Let denote $t \mapsto \theta_t(x_0)$ the solution of (1.0.2), so that $\theta_0 = \text{id}$ and $\theta_t(\theta_s(x_0)) = \theta_{t+s}(x_0)$. When $t$ varies on $\mathbb{R}$, the family $(\theta_t)_{t \in \mathbb{R}}$ is a group of transformations of $\Sigma$, known as the (classical) flow of solutions. The orbit of an element $x_0 \in \Sigma$ is $\theta(x) = (\theta_t(x_0))_{t \in \mathbb{R}}$.

We will construct a different representation of our system, which will prepare notations for the sequel. Call $\Omega = D([0, \infty[ \times \Sigma)$ the space of functions $\omega = (\omega(t); \ t \geq 0)$ defined in $[0, \infty[$
with values in $\Sigma$, which have left-hand limits ($\omega(t-) = \lim_{s \to t, s < t} \omega(s)$) and are right-continuous ($\omega(t+) = \lim_{s \to t, s > t} \omega(s) = \omega(t)$) on each $t \geq 0$ (with the convention $\omega(0-) = \omega(0)$).

Define $X_t(\omega) = \omega(t)$ for all $t \geq 0$. So that for each trajectory $\omega \in \Omega$, and any time $t \geq 0$, $X_t(\omega)$ is the state of the system at time $t$ when it follows the trajectory $\omega$. We can write $X_t(\omega) = (Q_t(\omega), P_t(\omega))$, where $Q_t, P_t : \Omega \to \mathbb{R}^3$ represent respectively, the position and momentum applications. These are particular examples of stochastic processes.

Thus, (1.0.2) may be written

$$dX_t(\omega) = J \nabla H(X_t(\omega)) dt, \quad X_0(\omega) = x_0.$$  

There is no great change in this writing of the equations of motion, however let us agree that such expression is a short way of writing an integral equation that is

$$X_t(\omega) = x_0 + \int_0^t J \nabla H(X_s(\omega)) ds.$$  

Solutions of the above equation are obviously continuous and differentiable. Moreover they preserve the total energy of the system:

$$H(X_t(\omega)) = H(x_0),$$  

for all $t \geq 0$.

### 1.2 The algebraic flow

The set of bounded continuous functions (one may consider also bounded measurable functions) from $\Sigma$ to $\mathbb{C}$ (respectively, from $\Omega$ to $\mathbb{C}$), constitute an algebra $\mathfrak{A}$, (resp. $\mathfrak{B}$). These algebras are endowed with an involution operation $^*$ which associates to each function $f$ its conjugate $\bar{f}$.

We define the algebraic flow associated to (1.0.2) by the map $j_t : \mathfrak{A} \to \mathfrak{B}$ given by

$$j_t(f)(\omega) = f(X_t(\omega)), \quad (t \in \mathbb{R}, \omega \in \Omega, f \in \mathfrak{A}).$$  

Notice that $j_t(f)$ satisfies a differential equation as well, since by the chain rule for any differentiable function $f$:

$$dj_t(f) = \langle \nabla f(X_t), dX_t \rangle = \langle \nabla f(X_t), J \nabla H(X_t) \rangle dt = \{H, f\}(X_t) dt,$$

where $\{u, v\} = \langle \nabla u, J \nabla v \rangle = \sum_{i=1}^3 \left( \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$ for two differentiable functions $u, v : \Sigma \to \mathbb{C}$ is the so-called Poisson-bracket.

So that, the equation of the flow, for all differentiable function $f$, can be written as

$$dj_t(f) = j_t(\{H, f\}) dt, \quad j_0 = \text{id}.$$  

(1.2.2)
For each \( x \in \Sigma \), the orbit \( \theta(x) \) is -as we explained- the solution of (1.0.2) with starting point \( x \). Call \( \delta_{\theta(x)} \) the Dirac measure supported by \( \{ \theta(x) \} \), so that \( \delta_{\theta(x)}(A) = 1 \) if and only if \( \theta(x) \in A \).

On the algebra \( \mathfrak{A} \) introduced before, we can define
\[
T_t f(x) = \int_\Omega j_t(f)(\omega)\delta_{\theta(x)}(d\omega) = \int_\Omega f(X_t(\omega))\delta_{\theta(x)}(d\omega),
\]
for all \( t \in \mathbb{R}, f \in \mathfrak{A} \) and \( x \in \Sigma \).

Notice that for each \( s, t \in \mathbb{R}, T_t(T_s(f)) = T_{t+s}(f), T_0(f) = f \), so that \( (T_t)_{t \in \mathbb{R}} \) is a group. This property is due to the fact that the system is conservative so that for a given \( t \geq 0 \), \( T_{-t} \) is the inverse of \( T_t \), which is characteristic of reversibility. In which follows we will introduce dissipation in our model so that the energy will not be conserved and the system will become irreversible. Thus, instead of taking time running over all the real numbers, we will consider \( t \in \mathbb{R}^+ \). In this case \( (T_t)_{t \in \mathbb{R}^+} \) is no more a group, but a semigroup of maps acting on \( \mathfrak{A} \).

This semigroup admits a generator \( L \) defined as
\[
\lim_{t \to 0} \frac{1}{t} (T_t f - f),
\]
for all \( f \) for which this limit exists (in the pointwise sense, for instance). That is,
\[
T_t f(x) = f(x) + \int_0^t L(T_s f(x))ds.
\]
A way to formally recover the above expression is
\[
Lf(x) = \frac{dT_t}{dt} f(x)|_{t=0} = \int_\Omega \frac{dt}{dt} f(X_t)|_{t=0}\delta_{\theta(x)}(d\omega),
\]
that is,
\[
Lf(x) = \{ H, f \}(x),
\]
for all \( x \in \Sigma \) whenever we take as domain of the generator the set \( D(L) \) of all \( C^1 \)-functions with bounded derivatives.

### 1.4 Opening the main system

Our basic space of trajectories \( \Omega \) allows discontinuities. Thus, we may modify our simple model by introducing kicks. Assume for instance that at a given time \( t_0 \) the particle collides with another object which introduces an instantaneous modification (force) on the momentum. Mathematically that variation on the momentum is given by a jump at time \( t_0 \), that is \( \Delta P_{t_0}(\omega) = P_{t_0}(\omega) - P_{t_0-}(\omega) \). From the physical point of view, we have changed the system: we no more have a single particle but a two-particle system. In the new system the jump in the momentum of the first particle is \((-1)\times\) the jump in the momentum of the second particle via the law of conservation of the momentum. Suppose that the magnitude of the jump in the momentum of the colliding particle (the instantaneous force) is \( c > 0 \), and call \( \xi(\omega) \) its sign, that is \( \xi(\omega) = 1 \) if the main particle is pushed forward, \( \xi(\omega) = -1 \) if it is pushed backwards. We then have
\[
\Delta P_{t_0}(\omega) = \xi(\omega)c = c\Delta V_{t_0}(\omega),
\]
where $V_t(\omega) = \xi(\omega)1_{[t_0, \infty)}(t)$ and $1_{[t_0, \infty]}$ is the characteristic function of $[t_0, \infty)$ (or the Heaviside function at $t_0$). The function $t \mapsto V_t(\omega)$ has finite variations on bounded intervals of the real line. Integration with respect to $V$ corresponds to the customary Lebesgue-Stieltjes theory which turns out to be rather elementary in this case: if $f$ is a right-continuous function,

$$F(t) = \int_{[0,t]} f(s) dV_s(\omega) = \sum_{0 < s \leq t} f(s) \Delta V_s(\omega),$$

which allows to use the short-hand writing $dF = f(t) dV(t)$. Thus the equation of motion is written simply

$$dX_t(\omega) = J\nabla H(X_t(\omega)) dt + \sigma(X_t) dV_t, \quad X_0(\omega) = x_0,$$

where

$$\sigma(x) = \begin{pmatrix} 0 \\ c \end{pmatrix},$$

$$dP_t(\omega) = c dV_t(\omega).$$

More generally, we let assume that $\sigma$ is a function of the state of the system, and denote by $K(x)$ a (primitive) function such that $\sigma(x) = J\nabla K(x)$, for instance $K(x) = \begin{pmatrix} 1 \\ -cq \end{pmatrix}$. This yields to

$$dX_t(\omega) = J\nabla H(X_t(\omega)) dt + J\nabla K(X_t) dV_t, \quad X_0(\omega) = x_0. \quad (1.4.1)$$

Take $h > 0$ and consider times $T^h_n = nh$. We suppose that a sequence of impulses takes place at times $T^h_1(\omega) < T^h_2(\omega) < \ldots < T^h_n(\omega) < \ldots$. Then, the process $V$ becomes

$$V^h_t(\omega) = \sum_{n=0}^{\infty} \xi_n(\omega) 1_{[T_n^h(\omega), T_{n+1}^h(\omega)]}(t) = \sum_{n=0}^{\lfloor t/h \rfloor} \xi_n,$$

where the sequence $(\xi_n(\omega))_{n \in \mathbb{N}}$ takes values in $\{-1, 1\}$.

Assume that the masses of the colliding particles are all identical to $c_h$. The energy dissipated during the collisions will be proportional to

$$c_h^2 \sum_{n=1}^{\lfloor t/h \rfloor} |\xi_n(\omega)|^2 = c_h^2 \lfloor t/h \rfloor,$$

since $|\xi_n(\omega)|^2 = 1$. To keep the dissipated energy finite as $h \to 0$, we need to choose $c_h$ proportional to $\sqrt{h}$. Let us examine what happens to $X_t$, which we denote $X^h_t$ to underline the dependence on $h$. Notice that

$$X^h_t(\omega) = X^h_0(\omega) + \int_0^t J\nabla H(X^h_s(\omega)) ds + \begin{pmatrix} 0 \\ c_h \end{pmatrix} \sqrt{h} V^h_t(\omega)$$

$$= X^h_0(\omega) + \int_0^t J\nabla H(X^h_s(\omega)) ds + \begin{pmatrix} 0 \\ c_h \end{pmatrix} \sqrt{h} \sum_{n=0}^{\lfloor t/h \rfloor} \xi_n(\omega).$$

Now we are faced to the following problem: from one hand, the dissipated energy is $h[t/h]$ which tends to $t$ if $h \to 0$; but we currently have no tools to prove that $X^h_t$ converges. To cope with this problem we need to modify the mathematical framework of our study by introducing probabilities.
1.5 Introducing probabilities

Consider the space Ω introduced before, endowed with the sigma-algebra \( \mathcal{F} \) generated by its open subsets, the Borel sigma-algebra.

To solve the limit problem stated in the previous section, we consider a probability measure \( \mathbb{P} \) for which the sequence \((\xi_n)_{n \in \mathbb{N}}\) satisfies:

- \( \xi_n \) is \( \mathbb{P} \)-independent of \( \xi_m \) for all \( n, m \);
- \( \mathbb{P}(\xi_n = \pm 1) = \frac{1}{2} \) for all \( n \).

Under these hypothesis we obtain that the characteristic function, or Fourier transform of \( M^h_t = \sqrt{h} V^h_t \) is

\[
\mathbb{E}(e^{iuM^h_t}) = \prod_{i=1}^{[t/h]} \mathbb{E}(e^{iu\sqrt{h}\xi_i}) = \left( \cos(u\sqrt{h}) \right)^{[t/h]}.
\]

The last expression is equivalent to \( (1 - u^2 h/2)^{[t/h]} \) as \( h \to 0 \), thus

\[
\lim_{h \to 0} \mathbb{E}(e^{iuM^h_t}) = e^{-u^2 t/2}.
\]

In Classical Probability Theory, the above result is known as the Central Limit Theorem, for the random variables \( M^h_t \): they converge in distribution towards a normal (or Gaussian) random variable with zero mean and variance \( t \).

However, that result can be improved.

We concentrate on the equation satisfied by the algebraic flow. If the trajectories satisfy the equations

\[
dX^h_t = J \nabla H(X^h_t)dt + J \nabla K(X^h_t) dM^h_t,
\]

the flow \( j^h_t(f) = f(X^h_t) \) satisfies

\[
dj^h_t(f) = j^h_t(ad_H(f))dt + j^h_t(ad_K(f))dM^h_t + j^h_t \left( \frac{1}{2} ad_K^2(f) \right) (dM^h_t dM^h_t - dt),
\]

where \( ad_H(f) = \{H, f\} \) the Poisson bracket, and \( ad_K^2(\cdot) = \{K, \{K, \cdot\}\} \). This can be rewritten in the form

\[
j^h_t(f) = j^h_t(Lf)dt + j^h_t(ad_K(f))dM^h_t + j^h_t \left( \frac{1}{2} ad_K^2(f) \right) (dM^h_t dM^h_t - dt),
\]

where,

\[
Lf = \frac{1}{2} ad_K^2(f) + ad_H(f).
\]

A mathematical model for the interaction of the main system with the environment is called a martingale. Martingales are usually taught in probability courses. We notice that in our example, the processes \( M^h \) are square integrable martingales with respect to the family of \( \sigma \)-algebras \( \mathcal{F}^h_t \) generated by the variables \( \xi_k, k \leq [t/h] \). We recall that to each square integrable martingale \( M \) one can associate a unique predictable increasing process \( A \) such that \( M^2 - A \) is a martingale. The Brownian motion is characterized by its associated increasing process: it is a continuous martingale for which the associated process is \( A_t = t \). Thus, as a shortcut, we denote \( dM_t dM_t \) the measure \( dA_t \), and we say that the Itô table \( dM^h_t dM^h_t \) converges to \( dA_t \) if the process \( A^h_t \) converges to \( A_t \). Now we can allow \( M^h_t \) to be a general family of square integrable martingales.
Theorem 1.5.1. Assume that for all $\epsilon > 0$, $t \geq 0$ it holds

$$E\left(\sum_{s \leq t} |\Delta M^h_s|^2 1_{|\Delta M^h_s| > \epsilon}\right) \to 0$$

as $h \to 0$, then the Itô table $dM^h_t dM^h_t$ converges in probability to $dt$ if and only if the processes $M^h$ converge in distribution towards a Brownian Motion.

This theorem is a particular version of the Central Limit Theorem for Martingales [46].

We thus obtain that the limit equation for the trajectories is of the form

$$dX_t = J\nabla H(X_t)dt + J\nabla K(X_t)dW_t, \ X_0 = x. \quad (1.5.6)$$

While that of the algebraic flow is

$$dj_t(f) = j_t(Lf)dt + j_t(ad_K(f))dW_t, \quad (1.5.7)$$

where $Lf$ is given by (1.5.5).

The semigroup which corresponds to this dynamics is given by

$$T_t f(x) = E(j_t(f)|X_0 = x). \quad (1.5.8)$$

And its generator is $L$.

To summarize, the dynamics of the open system is represented by a semigroup $(T_t)_{t \geq 0}$ acting on an algebra $A$ of bounded functions defined on $\Sigma$. Previously, we dilated, the original phase space introducing the space of trajectories $\Omega$. Probabilities allowed to consider interactions of the main system with the environment, which contains non observed entities.
Chapter 2

An algebraic view on Probability

We now move to Quantum Theory. Firstly, we write the dynamics of a closed system. Consider a complex separable Hilbert space \( \mathfrak{h} \): observables are self-adjoint elements of the algebra \( \mathcal{B}(\mathfrak{h}) \) of all linear bounded operators on \( \mathfrak{h} \), states are assimilated to density matrices, or positive trace-class operators \( \rho \) with \( \text{tr}(\rho) = 1 \).

2.1 The basic closed quantum dynamics

The dynamics is given by a group of unitary operators \( (U_t)_{t \in \mathbb{R}} \). Assume for simplicity that \( U_t = e^{-iHt} \) with \( H = H^* \in \mathcal{B}(\mathfrak{h}) \).

In this case, the evolution equation is simply:

\[
    dU_t = -iHU_t dt. \tag{2.1.1}
\]

The flow is the group of automorphisms \( j_t : \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h}) \) given by

\[
    j_t(x) = U_t^* x U_t, \quad (x \in \mathcal{B}(\mathfrak{h})). \tag{2.1.2}
\]

The equation of this flow is

\[
    dj_t(x) = j_t(i[H, x]) dt, \tag{2.1.3}
\]

and we define the semigroup as \( T_t(x) = j_t(x) \). So that its generator is given by the application \( \delta : \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h}) \) defined by

\[
    \delta(x) = i[H, x], \quad (x \in \mathcal{B}(\mathfrak{h})), \tag{2.1.4}
\]

where \( [H, x] = Hx - xH \) is the commutator.

The challenge is to provide a mathematical framework where one can include the formalism of quantum Mechanics and that of classical Probability Theory. This is required to properly speak about quantum open systems.
2.2 Algebraic probability spaces

An algebra $\mathfrak{A}$ on the complex field $\mathbb{C}$ is a vector space endowed with a product, $(a, b) \in \mathfrak{A} \times \mathfrak{A} \mapsto ab \in \mathfrak{A}$, such that

1. $a(b + c) = ab + ac,$
2. $a(\beta b) = \beta(ab) = (\beta a)b,$
3. $a(bc) = (ab)c,$

for all $a, b, c \in \mathfrak{A}, \beta \in \mathbb{C}.$

**Definition 2.2.1.** A $^\ast$-algebra is an algebra $\mathfrak{A}$ on the complex field $\mathbb{C}$ endowed with an involution $^\ast : \mathfrak{A} \to \mathfrak{A}$ such that

1. $(\alpha a + \beta b)^\ast = \bar{\alpha}a^\ast + \bar{\beta}b^\ast,$
2. $(a^\ast)^\ast = a,$
3. $(ab)^\ast = b^\ast a^\ast,$

for all $a, b \in \mathfrak{A}, \alpha, \beta \in \mathbb{C}.$ Elements of the form $a = b^\ast b$ are called positive, they form the cone of positive elements denoted by $\mathfrak{A}^+$. This cone introduces a partial order on the algebra: $a \leq b$ if $b - a \in \mathfrak{A}^+$, for all $a, b \in \mathfrak{A}.$

A $^\ast$-algebra $\mathfrak{D}$ satisfies Daniell’s condition, equivalently we say it is a $D^\ast$-algebra, if it contains a unit $1$; for any $a \in \mathfrak{D}^+$ there exists $\lambda > 0$ such that $a \leq \lambda 1$, and any increasing net $(a_\alpha)_{\alpha \in I}$ of positive elements with an upper bound in $\mathfrak{D}^+$ has a least upper bound $\sup_{\alpha \in I} a_\alpha$ in $\mathfrak{D}^+.$

An **algebraic probability space** is a couple $(\mathfrak{A}, E)$ where $\mathfrak{A}$ is a $^\ast$-algebra on the complex field endowed with a unit $1$ and $E : \mathfrak{A} \to \mathbb{C}$ is a linear form, called a state, such that

(S1) $E(a^\ast a) \geq 0$, for all $a \in \mathfrak{A}$ ($E(\cdot)$ is positive),

(S2) $E(1) = 1$.

We denote $S(\mathfrak{A})$ the convex set of all states defined over the algebra $\mathfrak{A}$.

Given another $^\ast$-algebra $\mathfrak{B}$, a **random variable** on $\mathfrak{A}$ with values on $\mathfrak{B}$ is a $^\ast$-homomorphism $j : \mathfrak{B} \to \mathfrak{A}.$

Such a random variable, defines an image-state on $\mathfrak{B}$, the law of $j$, by $E_j(B) = E(j(B))$, $B \in \mathfrak{B}$.

This is the more general setting in which essential definitions for a Probability Theory can be given. As it is, one can hardly obtain interesting properties unless further conditions on both the involved $^\ast$-algebras and states being assumed. Let us show first that the classical case is well included in this new theoretical framework.

**Example 1.** Given a measurable space $(\Omega, \mathcal{F})$, consider the algebra $\mathfrak{A} = b\mathcal{F}$ of all bounded measurable complex functions. This is a $D^\ast$-algebra. The corresponding algebraic probability space is then $(\Omega, E)$ where $E(X) = \int_{\Omega} X(\omega)d\mathbb{P}(\omega)$, for each $X \in \mathfrak{A}$, $\mathbb{P}$ being a probability measure on $(\Omega, \mathcal{F})$. 
Moreover, to a classical complex-valued random variable $X \in \mathfrak{A}$ corresponds an algebraic random variable as follows. Consider the algebra $\mathfrak{B}$ of all bounded borelian functions $f$ and define $j_X(f) = f(X)$. The map

$$j_X : \mathfrak{B} \to \mathfrak{A},$$

is clearly a *-homomorphism. In this case, the law $E_{j_X}(f) = E(f(X))$, defined on $\mathfrak{B}$, determines a measure on $\mathbb{C}$ endowed with its borelian $\sigma$-algebra which is the classical distribution of the random variable $X$.

**Definition 2.2.2.** Given an algebraic probability space $(\mathfrak{A},E)$, the state $E$ is normal if for any increasing net $(x_\alpha)_{\alpha \in I}$ of $\mathfrak{A}^+$ with least upper bound $\sup_\alpha x_\alpha$ in $\mathfrak{A}$ it holds

$$E\left(\sup_\alpha x_\alpha\right) = \sup_\alpha E(x_\alpha).$$

We denote $S_n(\mathfrak{A})$ the set of all normal states on the algebra $\mathfrak{A}$. A pure state is an element $E \in S_n(\mathfrak{A})$ for which the only positive linear functionals majorized by $E$ are of the form $\lambda E$ with $0 \leq \lambda \leq 1$.

Any projection $p \in \mathfrak{A}$, that is $p^2 = p$, is called an event. However, the set of projections in $\mathfrak{A}$ could be rather poor and in some cases reduced to the trivial elements 0 and 1.

**Proposition 2.2.1.** Given a *-algebra $\mathfrak{A}$, pure states are the extremal points of the convex set $S_n(\mathfrak{A})$.

**Proof.** Let $E$ be a pure state and suppose that $E = \lambda E_1 + (1 - \lambda)E_2$ with $E_i \in S_n(\mathfrak{A})$, $(i = 1, 2)$, and $0 < \lambda < 1$. Then $E_1 \leq E$ contradicting that $E$ is a pure state. Thus $E$ is an extremal point of the convex set $S_n(\mathfrak{A})$.

Now, if $E$ is extremal, let suppose that there exists a non trivial positive linear functional $\varphi \leq E$. We may assume $0 < \varphi(1) < 1$, otherwise we replace $\varphi$ by $\lambda \varphi$ with $0 < \lambda < 1$. Define

$$E_1 = \frac{1}{\varphi(1)} \varphi, \quad E_2 = \frac{1}{1 - \varphi(1)} (E - \varphi).$$

Both, $E_1$ and $E_2$ are states and $E = \varphi(1)E_1 + (1 - \varphi(1))E_2$. This is a contradiction since $E$ is extremal. Thus $E$ is a pure state. \qed

Before going on, let us say a word about the notation of states. In Probability the notation $E$ is more appealing, however in the tradition of operator algebras, states are oftenly denoted by greek letters like $\omega$. We will use both notations depending on the kind of properties we wish to emphasize.

In the previous example, $\mathfrak{A}$ contained non trivial events: all elements $p = 1_E$, with $E \in \mathfrak{S}$. We will see later a commutative algebraic probability space which has no nontrivial projections, but we first give a prototype of a non-commutative probability space.

**Example 2.** Consider the algebra $\mathfrak{A} = \mathfrak{M}_n(\mathbb{C}^n)$ the space of $n \times n$-matrices acting on the space $\mathfrak{h} = \mathbb{C}^n$. Given a positive density matrix with unit trace $\rho$, one defines a state as $E(A) = \text{tr}(\rho A)$. Thus, $(\mathfrak{A}, E)$ is an example of a non-commutative algebraic probability space.

An observable here is any self-adjoint operator $X$. Take $\mathfrak{B} = \mathfrak{A}$, and $U$ a unitary transformation of $\mathbb{C}^n$. Then $j(B) = U^*BU$, $B \in \mathfrak{B}$ defines a random variable.

In this case, $\mathfrak{A}$ has non-trivial events: any projection defined on $\mathbb{C}^n$.
Example 3. Consider in particular $\mathfrak{h} = \mathbb{C}^2$, with the canonical basis $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the algebra $\mathfrak{A} = \mathfrak{M}_n(\mathbb{C}^2)$. Any $2 \times 2$ matrix can be expressed as a linear combination of the self-adjoint Pauli spin matrices:

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(2.2.1)

Consider the pure state $E_0(x) = \text{tr}(|e_0\rangle\langle e_0|) = \langle e_0, xe_0 \rangle$, ($x \in \mathfrak{A}$). In this state, the observables $\sigma_1$ and $\sigma_2$ assume the value 1 and $-1$ with equal probability $E_0(\sigma_i) = 1/2$, ($i = 1, 2$), where $1_0(x)$ is the projection of an observable $x$ on the space generated by $e_0$. You may compare these observables with the random variables $\xi_n$ introduced in 1.5. Notice that $\sigma_3$ assumes the value 1 with probability $1$.

Example 4. Consider a compact space $\Omega$ endowed with its borelian $\sigma$-algebra $\mathfrak{B}(\Omega)$ and a Radon probability measure $\mathbb{P}$ (or Radon expectation $\mathbb{E} (\cdot)$). Now, take $\mathfrak{A} = C(\Omega, \mathbb{C})$ the algebra of continuous complex-valued functions. The space $(\mathfrak{A}, \mathbb{E})$ is a commutative probability space with no nontrivial events. This is an important space which is frequently used when studying classical dynamical systems.

The above is an example of an important class of $^*$-algebras, the class of $C^*$-algebras.

**Definition 2.2.3.** A $^*$-algebra $\mathfrak{A}$ endowed with a norm $\|\cdot\|$ is a Banach $^*$-algebra if it is complete with respect to the topology defined by this norm, which is referred to as the uniform topology, and $\|a\| = \|a^*\|$, for all $a \in \mathfrak{A}$. A Banach $^*$-algebra is a $C^*$-algebra if moreover,

$$
\|a^*a\| = \|a\|^2,
$$

(2.2.2)

for all $a \in \mathfrak{A}$.

A subspace $S$ of a unital $C^*$-algebra, is called an operator system if for any $s \in S$ one has $s^* \in S$ and $1 \in S$.

A von Neumann algebra on a Hilbert space $\mathfrak{h}$ is a $^*$-subalgebra $\mathfrak{M}$ of $\mathfrak{B}(\mathfrak{h})$ which is weakly closed. This is equivalent to $\mathfrak{M} = \mathfrak{M}''$ where the right hand term denotes the bicommutant. Another equivalent characterization is that $\mathfrak{M}$ is the dual of a Banach space denoted $\mathfrak{M}_*$ and called its predual.

Now we follow Hora and Obata [28] to introduce an important concrete class of algebraic probability spaces, which have been used in the spectral analysis of graphs. Some particular cases (boson and fermion cases) were introduced in Physics during the second half of twentieth century.

**Definition 2.2.4.** A sequence $(\omega_n)_{n \geq 1}$ of positive real numbers is called a Jacobi sequence if one of the following two conditions is satisfied:

\begin{align*}
&\text{(J1) Infinite type: } \omega_n > 0 \text{ for all } n; \\
&\text{(J2) Finite type: there exists a number } m_0 \geq 1 \text{ such that } \omega_n = 0 \text{ for all } n \geq m_0 \text{ and } \omega_n > 0 \text{ for all } n < m_0.
\end{align*}

By definition $(0, 0, 0, \ldots)$ is a Jacobi sequence. Any finite sequence of positive numbers can be identified with a finite type Jacobi sequence by concatenating an infinite sequence consisting of only zero.
Example 5. Consider an infinite-dimensional complex separable Hilbert space \( \mathfrak{h} \), endowed with a complete orthonormal basis \( (e_n)_{n \in \mathbb{N}} \). Let denote \( \mathfrak{h}_0 \) the dense linear subspace spanned by the orthonormal basis. Given a Jacobi sequence \( (\omega_n)_{n \geq 1} \) we associate the following linear operators in \( L(\mathfrak{h}_0) \), the vector space of all linear operators defined on \( \mathfrak{h}_0 \):

\[
\begin{align*}
a \dagger e_n & = \sqrt{\omega_{n+1}} e_{n+1}, \quad (n \geq 0) \quad (2.2.3) \\
ac_0 & = 0 \quad (2.2.4) \\
a e_n & = \sqrt{\omega_n} e_{n-1}, \quad (n \geq 1) \quad (2.2.5) \\
N e_n & = n e_n, \quad (n \geq 0). \quad (2.2.6)
\end{align*}
\]

It is immediately verified that \( a \dagger \) y a are mutually adjoint.

Moreover, let denote \( \Gamma \subset \mathfrak{h}_0 \) the linear space spanned by \( \{(a \dagger)^n e_0 : n \geq 0\} \). This space remain invariant under the action of both \( a \dagger \) and a.

**Definition 2.2.5.** The quadruple \( \Gamma_{(\omega_n)} = (\Gamma, (e_n), a \dagger, a) \) is called an *interacting Fock space* associated with a Jacobi sequence \( (\omega_n) \); \( e_0 \) is called the *vacuum vector* and \( e_n \), the *number vector*. The application \( a \dagger \) is called the *creation operator*; \( a \), the *annihilation operator* and \( N \), the *number operator*.

The algebraic probability space \( (L(\Gamma), E_0) \), where \( L(\Gamma) \) is the algebra of all linear operators defined on the linear space \( \Gamma \) and \( E_0(x) = \langle e_0, xe_0 \rangle \) \( (x \in L(\Gamma)) \) is called the *interacting Fock probability space* associated with a Jacobi sequence \( (\omega_n)_{n \geq 1} \).

The following relations are easily checked from the definitions:

\[
a \dagger a = \omega_N, \quad \text{(where we assume } \omega_0 = 0) \quad a a \dagger = \omega_{N+1}. \quad (2.2.7)
\]

**Example 6.** Consider the Jacobi sequence \( \omega_n = n, \quad n \geq 1 \). The interacting Fock space associated with this sequence is the *Boson Fock space*. Then (2.2.7) becomes

\[
[a, a \dagger] = aa \dagger - a \dagger a = 1
\]

referred to as the *Canonical Commutation Relation* (CCR).

**Example 7.** If one takes \( \omega_1 = 1 \) and \( \omega_n = 0 \) for all \( n > 1 \), one obtains the *Fermion Fock Space*, and

\[
\{a, a \dagger\} = aa \dagger + a \dagger a = 1,
\]

is referred as the *Canonical Anticommutation Relation* (CAR).
CHAPTER 2. AN ALGEBRAIC VIEW ON PROBABILITY
Chapter 3

Completely positive and completely bounded maps

3.1 From transition kernels to completely positive maps

We start by extending the classical notion of a transition kernel in Probability Theory. Let be given two measurable spaces \((E_i, \mathcal{E}_i), (i = a, b)\), and a kernel \(P(x, dy)\) from \(E_b\) to \(E_a\). That is, \(P : E_b \times \mathcal{E}_a\) is such that

- \(x \mapsto P(x, A)\) is measurable from \(E_b\) in \([0, 1]\) for any \(A \in \mathcal{E}_a\);
- \(A \mapsto P(x, A)\) is a probability on \((E_a, \mathcal{E}_a)\) for all \(x \in E_b\).

We denote \(A\) (respectively \(B\)) the algebra of all complex bounded measurable functions defined on \(E_a\) (resp. \(E_b\)). These are \(*\)-algebras (they have an involution given by the operation of complex conjugation) with unit. Moreover, they are \(C^*\)-algebras since they are complete for the topology defined by the uniform norm. The kernel \(P\) defines a linear map \(\Phi_P\) from \(A\) to \(B\) given by \(\Phi_P(a) = Pa\), where

\[ Pa(x) = \int_{E_b} P(x, dy)a(y), \]

for all \(a \in A, x \in E_b\).

It is worth noticing that \(\Phi_P\) is a positive map, moreover it satisfies a stronger property: for any finite collection of elements \(a_i \in A, b_i \in B, (i = 1, \ldots, n)\), the function

\[ \sum_{i,j=1}^{n} b_i \Phi_P(a_i \bar{a}_j) \bar{b}_j, \tag{3.1.1} \]

is positive. Indeed, for any fixed \(x \in E_b\), \(P(x, \cdot)\) is positive definite, so that for any collection \(\alpha_1, \ldots, \alpha_n\) of complex numbers, the sum

\[ \sum_{i,j} \alpha_i \bar{\alpha}_j P(a_i \bar{a}_j)(x), \]

is positive. Indeed, for any fixed \(x \in E_b\), \(P(x, \cdot)\) is positive definite, so that for any collection \(\alpha_1, \ldots, \alpha_n\) of complex numbers, the sum

\[ \sum_{i,j} \alpha_i \bar{\alpha}_j P(a_i \bar{a}_j)(x), \]
is positive. It is enough to choose \( \alpha_i = b_i(x), \) \((i = 1, \ldots, n)\), to obtain (3.1.1).

Now, take a probability measure \( \mu \) on \((E_b, E_a)\), call \( \Omega = E_b \times E_a, \mathcal{F} = E_b \otimes E_a \) and define a probability \( \mathbb{P} \) on \((\Omega, \mathcal{F})\) given by

\[
\mathbb{E}(b \otimes a) = \int_{E_b} \mu(dx)b(x)Pa(x),
\]

(3.1.2)

where \( a \in A, b \in B \).

Under the probability \( \mathbb{P} \), the random variables \((X_b, X_a)\), given by the coordinate maps on \(E_b \times E_a\), satisfy the following property: \( \mu \) is the distribution of \( X_b \) and \( \mathbb{P}(x, dy) \) is the conditional probability of \( X_a \) given that \( X_b = x \).

We now study the construction of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \). With this purpose consider the family of random variables of the form \( X = \sum_{i=1}^{n} b_i \otimes a_i \) with \( a_i \in A, b_i \in B, (i = 1, \ldots, n) \). The scalar product of two of such elements, is

\[
\langle X^{(1)}, X^{(2)} \rangle = \int_{E_b} \mu(dx) \sum_{i,j} b_i^{(1)*}(x)P(a_i^{(1)*}a_j^{(2)*})(x)b_j^{(2)*}(x).
\]

(3.1.3)

Notice that (3.1.1) is needed if one wants to define the scalar product through (3.1.3). Within this commutative framework, the property (3.1.1) is granted by the positivity of the kernel. This fails in the non-commutative case.

**Definition 3.1.1.** Let be given two \( \ast \)-algebras \( \mathfrak{A}, \mathfrak{B} \) and an operator system \( \mathcal{S} \) which is a subspace of \( \mathfrak{A} \). A linear map \( \Phi : \mathcal{S} \to \mathfrak{B} \) is completely positive if for any two finite collections \( a_1, \ldots, a_n \in \mathcal{S} \) and \( b_1, \ldots, b_n \in \mathfrak{B} \), the element

\[
\sum_{i,j=1}^{n} b_i^{*}\Phi(a_i^{*}a_j)b_j \in \mathfrak{B},
\]

is positive.

The set of all completely positive maps from \( \mathcal{S} \) to \( \mathfrak{B} \) is denoted \( \text{CP}(\mathcal{S}, \mathfrak{B}) \).

We restrict our attention to \( C^\ast \)-algebras and recall that a representation of a \( C^\ast \)-algebra \( \mathfrak{A} \) is a couple \((\pi, \mathfrak{h})\), where \( \mathfrak{h} \) is a complex Hilbert space and \( \pi \) is a \( \ast \)-homomorphism of \( \mathfrak{A} \) and the algebra of all bounded linear operators on \( \mathfrak{h}, \mathcal{B}(\mathfrak{h}) \).

**Remark 3.1.2.** Assume that the algebra \( \mathfrak{B} \) is included in \( \mathcal{B}(\mathfrak{h}) \) say, for a given complex separable Hilbert space \( \mathfrak{h} \). Then the positivity of the element

\[
\sum_{i,j=1}^{n} b_i^{*}\Phi(a_i^{*}a_j)b_j,
\]

introduced before is equivalent to

\[
\sum_{i,j=1}^{n} \langle u, b_i^{*}\Phi(a_i^{*}a_j)b_j u \rangle \geq 0,
\]

for all \( u \in \mathfrak{h} \). Equivalently,

\[
\sum_{i,j=1}^{n} \langle u_i, \Phi(a_i^{*}a_j)u_j \rangle \geq 0,
\]

(3.1.4)

for all collection of vectors \( u_1, \ldots, u_n \in \mathfrak{h} \).
3.1. FROM TRANSITION KERNELS TO COMPLETELY POSITIVE MAPS

Two complementary results, one due to Arveson and the second proved by Stinespring, show that complete positivity is always derived from positivity in the commutative case. More precisely,

**Theorem 3.1.3.** Given a positive map \( \Phi : A \to B \), it is completely positive if at least one of the two conditions below is satisfied

(a) \( A \) is commutative (Stinespring [52]);
(b) \( B \) is commutative (Arveson [4]).

**Proof.**

(a) Suppose \( B \subseteq B(\mathcal{H}) \) for a complex and separable Hilbert space \( \mathcal{H} \). Since \( A \) is a commutative \( C\ast \)-algebra containing a unit \( 1 \), it is isomorphic to the space of continuous functions defined on a compact Hausdorff space (the spectrum \( \sigma(A) \) of \( A \)). So that any element \( a \in A \) is identified with a continuous function \( a(x), x \in \sigma(A) \). Therefore, since the map \( \Phi \) is positive, linear, and \( \Phi(1) = 1 \), it follows that for all \( u, v \in \mathcal{H} \), there exists a complex-valued Baire measure with finite total variation \( \mu_{u,v} \) such that

\[
\langle v, \Phi(a)u \rangle = \int_{\sigma(A)} d\mu_{u,v}(x)a(x).
\]

Take now arbitrary vectors \( u_1, \ldots, u_n \in \mathcal{H} \). Define

\[
d\mu = \sum_{i,j} |d\mu_{u_i,u_j}|,
\]

where the vertical bars denote total variation of the corresponding measure. Then each measure \( \mu_{u_i,u_j} \) is absolutely continuous with respect to the positive measure \( \mu \). Let \( h_{u_i,u_j} \) denote the Radon-Nykodim derivative of \( \mu_{u_i,u_j} \). Put \( u = \sum_{i} \lambda_i u_i \), then

\[
d\mu_{u,u} = \left( \sum_{i,j} \bar{\lambda}_i \lambda_j h_{u_i,u_j} \right) d\mu,
\]

and since both measures \( \mu_{u,u} \) and \( \mu \) are positive, it follows that

\[
\sum_{i,j} \bar{\lambda}_i \lambda_j h_{u_i,u_j} \geq 0,
\]

\( \mu \)-almost surely for all finite collection \( \lambda_1, \ldots, \lambda_n \) of complex numbers. Furthermore, for any collection \( a_1, \ldots, a_n \in A \),

\[
\sum_{i,j} \langle u_i, \Phi(a_i^* a_j)u_j \rangle = \int_{\sigma(A)} d\mu(x) \left( \sum_{i,j} a_i(x)a_j(x)h_{u_i,u_j}(x) \right) \geq 0.
\]

Thus, \( \Phi \) is completely positive.
(b) If $\mathcal{B}$ is commutative, we identify elements $b$ of $\mathcal{B}$ with continuous functions $b(x)$. Thus, given arbitrary collections $a_1, \ldots, a_n \in \mathcal{A}$, $b_1, \ldots, b_n \in \mathcal{B}$,

$$\sum_{i,j} b_i(x) \Phi(a_i^* a_j)b_j(x) = \Phi\left(\left[\sum_k b_k(x) a_k\right]^* \left[\sum_k b_k(x) a_k\right]\right) \geq 0.$$  

Thus, the notion of complete positivity attains its full sense in the pure noncommutative framework, that is, when both $\mathcal{A}$ and $\mathcal{B}$ are non-abelian. For each $n \geq 1$, let denote $M_n(\mathcal{A})$ the algebra of all $n \times n$-matrices $(a_{i,j})$, where $a_{i,j} \in \mathcal{A}$. Moreover, to any linear map $\Phi : \mathcal{A} \to \mathcal{B}$, we associate the map $\Phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ defined by

$$\Phi_n ((a_{i,j})) = (\Phi(a_{i,j})).$$  

(3.1.5)

The following characterization follows immediately from the definition.

**Proposition 3.1.1.** Given two $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if $\Phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is positive for all $n \geq 1$.

**Definition 3.1.4.** A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is $n$-positive if $\Phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is positive.

As we will see in the next section, the study of a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ through the induced sequence of maps $(\Phi_n)$ is a powerful procedure. Especially because we can use well-known features about matrix algebra to obtain results for linear maps between $C^*$-algebras.

For instance, consider a Hilbert space $\mathfrak{h}$, positive operators $P, Q \in B(\mathfrak{h})$, and $A$ any bounded operator. Take $\lambda \in \mathbb{C}$, and vectors $u, v \in \mathfrak{h}$. Compute

$$\left\langle \begin{pmatrix} \lambda u \\ v \end{pmatrix}, \begin{pmatrix} P & A \\ A^* & Q \end{pmatrix} \begin{pmatrix} \lambda u \\ v \end{pmatrix} \right\rangle = \lambda^2 \langle u, Pu \rangle + \lambda \langle u, Av \rangle + \lambda \overline{\langle u, Av \rangle} + \langle v, Qv \rangle.$$  

Thus the right-hand term is positive if and only if

$$|\langle u, Av \rangle|^2 \leq \langle u, Pu \rangle \langle v, Qv \rangle.$$  

(3.1.6)

From this elementary computation we derive that the matrix

$$\begin{pmatrix} P & A \\ A^* & Q \end{pmatrix},$$  

is positive if and only if (3.1.6) holds. As a result we obtain:

**Proposition 3.1.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras. We assume that $\mathcal{A}$ has a unit.

1. Suppose $a \in \mathcal{A}$, then $\|a\| \leq 1$ if and only if the matrix

$$\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix},$$

is positive in $M_2(\mathcal{A})$.  


2. Let \( b \in \mathfrak{A} \) be a positive element of \( \mathfrak{A} \). Then \( a^*a \leq b \) if and only if the matrix

\[
\begin{pmatrix}
1 & a \\
\ast & b
\end{pmatrix}
\]

is positive in \( \mathcal{M}_2(\mathfrak{A}) \).

3. Suppose that \( \mathfrak{B} \) is also unital and that \( \Phi : \mathfrak{A} \to \mathfrak{B} \) is a 2-positive linear map which preserves the unit. Then \( \Phi \) is contractive.

4. Let \( \Phi \) be a unital 2-positive linear map as before. Then \( \Phi(a)^*\Phi(a) \leq \Phi(a^*a) \), for all \( a \in \mathfrak{A} \). This is known as the Schwartz inequality for 2-positive maps.

**Proof.**

1. Taking a representation \((\pi, \mathfrak{B})\) of \( \mathfrak{A} \), let \( A = \pi(a), P = Q = 1 \) in (3.1.7), which is positive if and only if \(|\langle u, Av \rangle|^2 \leq \|u\|^2\|v\|^2\) for all \( u,v \in \mathfrak{B} \). This is equivalent to the condition \( \|a\| \leq 1 \).

2. Similarly, choosing \( P = 1, Q = \pi(b), A = \pi(a) \), the positivity of the matrix (3.1.7) is equivalent to \(|\langle u, Av \rangle|^2 \leq \|u\|^2\langle u, Qv \rangle\), which holds if and only if \( \|a^*a\| = \|A\|^2 \leq \|Q^{1/2}\|^2 = \|b^{1/2}\|^2 \), that is, \( a^*a \leq b \).

3. Notice that for any \( a \in \mathfrak{A} \) such that \( \|a\| \leq 1 \),

\[
\Phi_2\begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix} = \begin{pmatrix} 1 & \Phi(a) \\ \Phi(a)^* & 1 \end{pmatrix}
\]

is positive, so that \( \|\Phi(a)\| \leq 1 \).

4. For any element \( a \in \mathfrak{A} \), the product

\[
\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ a^* & a^*a \end{pmatrix},
\]

is positive. Thus,

\[
\begin{pmatrix} 1 & \Phi(a) \\ \Phi(a)^* & \Phi(a^*a) \end{pmatrix} \geq 0.
\]

By part 2 before, we obtain that \( \Phi(a)^*\Phi(a) \leq \Phi(a^*a) \).

Let \( f \) be a finite-dimensional space. The algebra \( \mathcal{B}(f) \) is isomorphic to the algebra of \( n \times n \) matrices \( \mathcal{M}_n \) for some \( n \). Suppose that \( \mathcal{S} \) is an operator system contained in a C*-algebra \( \mathfrak{A} \) and let \( \Phi : \mathfrak{A} \to \mathcal{B}(f) \) be a completely positive map. An extension theorem due to Krein shows that any positive map defined on \( \mathcal{S} \) with values in \( \mathbb{C} \) can be extended to all of \( \mathfrak{A} \). So that, for all \( m \), the positive map \( \Phi_m : \mathcal{M}_m(\mathcal{S}) \to \mathcal{M}_n \) can be extended to \( \mathcal{M}_m(\mathfrak{A}) \). This means that the completely positive map \( \Phi : \mathcal{S} \to \mathcal{M}_n \) can be extended to all of \( \mathfrak{A} \) that is, there exists a CP map \( \Psi : \mathfrak{A} \to \mathcal{M}_n \) such that \( \Psi|_{\mathcal{S}} = \Phi \). The following crucial result proved by Arveson gives the main extension theorem for CP maps.
We say that $\Phi$ is $
abla$-positive if $\langle \Phi(u)u, v \rangle \geq 0$ for all $u, v \in \mathcal{A}$.

Theorem 3.1.5 (Arveson). Let $\mathfrak{A}$ be a $C^*$-algebra, $\mathcal{S}$ an operator system contained in $\mathfrak{A}$ and $\Phi : \mathcal{S} \to \mathcal{B}(\mathfrak{h})$ a completely positive map. Then there exists a completely positive map $\Psi : \mathfrak{A} \to \mathcal{B}(\mathfrak{h})$ which extends $\Phi$.

Proof. Consider the directed net $FD$ of all finite-dimensional subspaces $\mathfrak{f}$ of $\mathfrak{h}$. Denote $P_1$ the projection onto $\mathfrak{f}$ defined on $\mathfrak{h}$ and call $\Phi_i(a) = P_1\Phi(a)|_{\mathfrak{f}}$, $a \in \mathcal{S}$, the compression (or reduction) of $\Phi$ to $\mathfrak{f}$. From the previous discussion, we know that there exists a completely positive map $\Psi_i : \mathfrak{A} \to \mathcal{B}(\mathfrak{f})$ which extends $\Phi_i$, since $\mathcal{B}(\mathfrak{f})$ is isomorphic to an algebra of finite-dimensional matrices. Defining $\Psi$ to be 0 on the orthogonal complement of $\mathfrak{f}$, we extend the range of this map to $\mathcal{B}(\mathfrak{h})$. Moreover $\|\Phi_i\| \leq \|\Phi(1)\|$, for all $\mathfrak{f} \in FD$, so that this net is compact in the $w^*$-topology by an application of the Banach-Alaglou Theorem. As a result, there exists a limit point, a completely positive map $\Psi$, such that $\|\Psi\| \leq \|\Phi\|$. We prove that $\Psi$ extends $\Phi$. Indeed, let $a \in \mathcal{S}$, $u, v \in \mathfrak{h}$ and denote $\mathfrak{f}$ the vector space generated by $u$ and $v$. Then, for any other finite-dimensional subspace $\mathfrak{f}_1$ of $\mathfrak{h}$ which contains $\mathfrak{f}$ it holds $\langle v, \Phi(a)u \rangle = \langle v, \Psi_i(a)u \rangle$. Thus, since $\mathfrak{f}_1$ is cofinal, we obtain $\langle v, \Phi(a)u \rangle = \langle v, \Psi(a)u \rangle$. □

3.2 Completely bounded maps

The sum of completely positive maps is again completely positive as well as the composition of two of such maps. Furthermore, any $^*$-homomorphism of algebras is completely positive. Thus, given any representation $(\pi, \mathfrak{A})$ of the $C^*$-algebra $\mathfrak{A}$, $\pi$ is completely positive. To summarize, the set $\text{CP}(\mathfrak{A}, \mathfrak{B})$ of completely positive maps from $\mathfrak{A}$ to $\mathfrak{B}$ defines a cone.

In a $C^*$-algebra $\mathfrak{A}$, the cone $\mathfrak{A}^+$ of positive elements defines a norm-closed convex cone. If $h \in \mathfrak{A}$ is a self-adjoint element, the functional calculus shows easily that $h$ can be written as the difference of two positive elements. Indeed, it suffices to use the decomposition of any real number $x$ in its positive $x^+ = \sup \{x, 0\}$ and negative parts $x^- = \sup \{-x, 0\}$. Furthermore, using the Cartesian decomposition of an arbitrary element $a \in \mathfrak{A}$, namely, $a = h + ik$, where $h$ and $k$ are self-adjoints, one obtains

$$a = (h^+ - h^-) + i(k^+ - k^-),$$

where $h^\pm, k^\pm$ are positive elements of $\mathfrak{A}$. Thus $\mathfrak{A}$ is the complex linear span of $\mathfrak{A}^+$.

We want to extend this property to $\text{CP}(\mathfrak{A}, \mathfrak{B})$, for two $C^*$-algebras. That is, we want to study the complex linear span of the above cone.

Definition 3.2.1. With the notations previous to Proposition 3.1.1, let $\Phi : \mathfrak{A} \to \mathfrak{B}$ be a linear map. We say that $\Phi$ is completely bounded if $\|\Phi\|_{\text{cb}} := \sup_n \|\Phi_n\| < \infty$. The normed space of completely bounded maps from $\mathfrak{A}$ to $\mathfrak{B}$ is denoted $\text{CB}(\mathfrak{A}, \mathfrak{B})$.

It is easily seen that $\text{CB}(\mathfrak{A}, \mathfrak{B})$ is indeed a Banach space and any $\Phi \in \text{CB}(\mathfrak{A}, \mathfrak{B})$ can be decomposed into a linear combination of completely positive maps. Indeed, this theory in its current development, has obtained deeper results which the interested reader can follow in the book of Paulsen [41]. We limit ourselves to give below a partial account of those important properties.

Proposition 3.2.2. Let $\mathcal{S}$ be an operator system in a $C^*$-algebra with unit and $\Phi : \mathcal{S} \to \mathfrak{B}$ a completely positive map, where $\mathfrak{B}$ is another $C^*$-algebra. Then $\Phi$ is completely bounded and $\|\Phi\|_{\text{cb}} = \|\Phi(1)\|$. 

Proof. It is clear that $\|\Phi(1)\| \leq \|\Phi\| = \|\Phi\|_{cb}$. So that we only need to prove that $\|\Phi\|_{cb} \leq \|\Phi(1)\|$. Denote $1_n$ the unit of $\mathcal{M}_n(\mathfrak{A})$. Let $A = (a_{i,j})$ be in $\mathcal{M}_n(\mathfrak{S})$ and $\|A\| \leq 1$. The matrix,

$$
\begin{pmatrix}
1_n & A \\
A^* & 1_n
\end{pmatrix},
$$

is positive, hence so is

$$
\Phi(1_n A A^* 1_n) = \begin{pmatrix}
\Phi_n(1_n) & \Phi_n(A) \\
\Phi_n(A)^* & \Phi_n(1_n)
\end{pmatrix},
$$

Therefore, $\|\Phi_n(A)\| \leq \|\Phi_n(1_n)\| = \|\Phi(1)\|$. □

Like in Theorem 3.1.3 we obtain that bounded maps are completely bounded if its range is an abelian $C^*$-algebra.

Theorem 3.3.1 (Stinespring). Let $\mathfrak{S}$ be an operator system and $\Phi : \mathfrak{S} \to \mathfrak{B}$ a bounded linear map, where $\mathfrak{B}$ is a commutative $C^*$-algebra. Then $\|\Phi\|_{cb} = \|\Phi\|$.

Proof. Since $\mathfrak{B}$ is commutative, we identify elements $b$ of $\mathfrak{B}$ with continuous functions $b(x)$ defined on a compact Hausdorff space $X$. Every element $B = (b_{i,j})$ of $\mathcal{M}_n(\mathfrak{B})$ is identified with continuous matrix-valued functions; multiplication is just pointwise multiplication and the involution is the $^*$ operation on matrices. $\mathcal{M}_n(\mathfrak{B})$ is a $C^*$-algebra with the norm $\|B\| = \sup \{\|B(x)\| : x \in X\}$.

Let $x \in X$, and define $\Phi^x : \mathfrak{S} \to \mathbb{C}$ by $\Phi^x(a) = \Phi(a)(x)$. Thus,

$$
\|\Phi_n\| = \sup \{\|\Phi_n^x\| : x \in X\} = \sup \{\|\Phi^x\| : x \in X\} = \|\Phi\|.
$$

□

3.3 Dilations of CP and CB maps

Throughout this section we assume that the $C^*$–algebras $\mathfrak{A}$ and $\mathfrak{B}$ have a unit denoted in both cases by the same symbol $1$.

Theorem 3.3.1 (Stinespring). Let $\mathfrak{B}$ be a sub $C^*$–algebra of the algebra of all bounded operators on a given complex separable Hilbert space $\mathfrak{h}$. Assume $\mathfrak{A}$ to be a $C^*$–algebra with unit. A linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is completely positive if and only if it has the form

$$
\Phi(x) = V^* \pi(x) V,
$$

where $(\pi, \xi)$ is a representation of $\mathfrak{A}$ on some Hilbert space $\mathfrak{k}$, and $V$ is a bounded operator from $\mathfrak{h} \to \mathfrak{k}$.

Proof. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$–algebras, with $\mathfrak{B} \subset B(\mathfrak{h})$, where $\mathfrak{h}$ is a complex separable Hilbert space. Let be given a completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$. Take two arbitrary elements $x = \sum_i a_i \otimes u_i$, $y = \sum_j b_j \otimes v_j$ in the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{h}$, where both sums contain a finite number of terms, and define

$$
\langle x, y \rangle = \sum_{i,j} \langle u_i, \Phi(a_i^* b_j) v_j \rangle.
$$
Since $\Phi$ is completely positive, $\langle x, x \rangle \geq 0$. Denote
\[ N = \{ x \in A \otimes h; \langle x, x \rangle = 0 \}, \]
and introduce on the quotient space $(A \otimes h)/N$ the scalar product
\[ \langle x + N, y + N \rangle = \langle x, y \rangle. \]
By completion, we obtain a Hilbert space denoted $k$.

Our purpose now is to define a $^*$-homomorphism $\pi : A \to B(k)$. This is done in two steps.
Firstly, define $\pi_0(a)$ for $a \in A$ on elements of the form $x$ before:
\[ \pi_0(a) \left( \sum_i a_i \otimes u_i \right) = \sum_i (aa_i) \otimes u_i. \]
For $x$ and $y$ as before, $\pi_0(a)$ is a linear application in $A \otimes H$ which satisfies
\[ \langle x, \pi_0(a) \rangle = \langle \pi_0(a^*), x \rangle \] (3.3.2)
\[ \|\pi_0(a)x\|^2 = \langle x, \pi_0(a^*a)x \rangle \leq \|a^*a\|\langle x, \pi_0(1)x \rangle \]
\[ \leq \|a\|^2\|x\|^2. \] (3.3.3)

From the above relations, $\pi_0$ extends into a $^*$-homomorphism $\pi : A \to B(k)$ and $(\pi, k)$ is a representation of $A$.
Moreover, we can define a linear operator $V : h \to k$ by
\[ Vu = 1 \otimes u + N. \]
This is a bounded operator since
\[ \|Vu\|^2 = \langle u, \Phi(1)u \rangle \leq \|\Phi(1)\|\|u\|^2. \]
Finally, $\Phi$ may be written in the form
\[ \Phi(a) = V^* \pi(a)V, \]
for all $a \in A$.

On the other hand, if $\Phi$ is given through (3.3.1), an elementary computation shows that $\Phi$ is completely positive. □

Thus we have obtained the celebrated characterization of completely positive maps due to Stinespring [52] (see also [38], [43]). The representation (3.3.1) is not unique. We call the couple $(\pi, V)$ a Stinespring representation of $\Phi$. Moreover, the above representation is said to be minimal if $\{ \pi(x)Vu = x \in A, u \in h \}$ is dense in $k$. For a given completely positive map, the minimal representation is unique up to a unitary equivalence.

**Proposition 3.3.1.** Let $\mathfrak{A}$ be a $C^*$-algebra and $\Phi : \mathfrak{A} \to B(h)$ a completely positive map. Suppose two minimal Stinespring dilations $(\pi_i, V_i, \mathfrak{k}_i)$, $i = 1, 2$, be given for $\Phi$. Then there exists a unitary operator $U : \mathfrak{k}_1 \to \mathfrak{k}_2$ which satisfies $UV_1 = V_2$ and $U \pi_1 U^* = \pi_2$. 
3.3. DILATIONS OF CP AND CB MAPS

Proof. Vectors like \( \sum_{j=1}^{n} \pi_i(a_j)V_1u_j \) form a dense subset \( \mathfrak{U}_i \) of \( \mathfrak{h}_i \), \((i = 1, 2)\). Thus, the theorem follows from mapping these two dense subsets via an operator \( U \). Define
\[
U \left( \sum_{j=1}^{n} \pi_1(a_j)V_1u_j \right) = \sum_{j=1}^{n} \pi_2(a_j)V_2u_j,
\]
for any integer \( n \geq 1 \), \( a_1, \ldots, a_n \in \mathfrak{A} \), \( u_1, \ldots, u_n \in \mathfrak{h} \). The density of \( \mathfrak{U}_1 \) and \( \mathfrak{U}_2 \) implies that \( U \) is onto. It remains to prove that it is an isometry, which follows from the computation below:
\[
\left\| \sum_{j=1}^{n} \pi_1(a_j)V_1u_j \right\|^2 = \sum_{i,j} \langle u_i, \pi_1(a_i^*a_j)V_1u_j \rangle
\]
\[
= \sum_{i,j} \langle u_i, \Phi(a_i^*a_j)u_j \rangle
\]
\[
= \left\| \sum_{j=1}^{n} \pi_2(a_j)V_2u_j \right\|^2.
\]

If the completely positive map \( \Phi \) is \( \sigma \)-weakly continuous and preserves the identity, then its minimal representation \((\pi, V)\) is such that \( \pi \) is \( \sigma \)-weakly continuous, and \( V \) is an isometry: \( V^*V = 1 \). We denote by \( \text{CP}(\mathfrak{A}, \mathfrak{B}) \) the set of all \( \sigma \)-weakly continuous completely positive maps \( \Phi : \mathfrak{A} \to \mathfrak{B} \) which preserve the identity. Furthermore, in this case \( \mathfrak{h} \) may be identified with the subspace \( V\mathfrak{h} \) of \( \mathfrak{h} \), \( V^* \) becoming the projection \( P_\mathfrak{h} \) onto this subspace and the representation of \( \Phi \) can be written
\[
\Phi(a) = P_\mathfrak{h} \pi(a)|_\mathfrak{h},
\]
for all \( a \in \mathfrak{A} \).

For a von Neumann algebra \( \mathfrak{A} \), and \( \mathfrak{B} = \mathcal{B}(\mathfrak{h}) \), Kraus (see [34]) obtained the following characterization of normal completely positive maps.

**Theorem 3.3.2** (Kraus). Let be given two complex separable Hilbert spaces \( \mathfrak{h} \), \( \mathfrak{k} \), and a von Neumann algebra \( \mathfrak{A} \) of operators of \( \mathfrak{h} \). Then a linear map \( \Phi : \mathfrak{A} \to \mathfrak{B}(\mathfrak{k}) \) is normal and completely positive if and only if there exists a sequence \((V_j)_{j \in \mathbb{N}}\) of linear bounded operators from \( \mathfrak{k} \) to \( \mathfrak{h} \) such that the series \( \sum_{j=1}^{\infty} V_j^*aV_j \) strongly converges for any \( a \in \mathfrak{A} \) and
\[
\Phi(a) = \sum_{j=1}^{\infty} V_j^*aV_j. \quad (3.3.4)
\]

**Proof.** It suffices to show that there exists a representation of a normal \( \pi \) in (3.3.1) leading to (3.3.4). Firstly, it can be shown that there exist a sequence of vectors \((u_n)_{n \in \mathbb{N}}\) in \( \mathfrak{h} \) such that \( \sum_{n} \|u_n\|^2 = 1 \) and \( \langle \Omega, \pi(a)\Omega \rangle = \sum_{n} \langle u_n, au_n \rangle \), where \( \Omega \) is a cyclic vector for \( \pi(\mathfrak{A}) \).

Moreover,
\[
\|xu_n\|^2 = \langle u_n, (x^*x)u_n \rangle \leq \langle \Omega, \pi(x^*x)\Omega \rangle = \|\pi(x)\Omega\|^2
\]
Let then, $V_n \pi(x) \Omega = xu_n$, for all $x \in \mathfrak{A}$. Thus we have,

$$\langle \pi(x) \Omega, \pi(a) \pi(x) \Omega \rangle = \sum_j \langle \pi(x) \Omega, V_j^* a V_j \pi(x) \Omega \rangle.$$

\[\square\]

**Remark 3.3.3.** The above representation can be improved by introducing an additional arbitrary complex and separable Hilbert space $\tilde{h}$ with an orthonormal basis $(f_n)_{n \in \mathbb{N}}$. Indeed, defining $V : \mathfrak{k} \to \mathfrak{h} \otimes \tilde{h}$ by

$$Vu = \sum_j V_j u \otimes f_j, \quad (u \in \mathfrak{k}),$$

then

$$\Phi(a) = V^*(a \otimes 1)V,$$

where $1$ is the identity operator of $\tilde{h}$, $a \in \mathfrak{A}$.

**Remark 3.3.4.** Following the same procedure used to prove the Kraus representation of a completely positive map $\Phi$, one can obtain a dilation based on random operators. Indeed, take $\Phi$ like in Theorem 3.3.4. Denote $E$ an orthonormal basis in $\mathfrak{h}$ (which is countable, since $\mathfrak{h}$ has been assumed separable). On the space $E$ define the $\sigma$-algebra of all subsets and define a probability $\mu$ such that

$$\langle \Omega, \pi(a) \Omega \rangle = \int_E \langle e, a e \rangle \mu(de) = \sum_{e \in E} \langle e, a e \rangle \mu(\{e\}) = \mathbb{E}(\langle \cdot, a \cdot \rangle),$$

where $\Omega$ is a cyclic vector for $\pi(\mathfrak{A})$.

Now define, like in the proof of 3.3.4, $V(e) \pi(x) \Omega = x e$, $x \in \mathfrak{A}$, $e \in E$, which yields,

$$\langle \pi(x) \Omega, \pi(a) \pi(x) \Omega \rangle = \sum_{e \in E} \mu(\{e\}) \langle \pi(x) \Omega, V^*(e) a V(e) \pi(x) \Omega \rangle$$

$$= \mathbb{E}(\langle \pi(x) \Omega, V^*(\cdot) a V(\cdot) \pi(x) \Omega \rangle)$$

$$= \langle \pi(x) \Omega, \mathbb{E}(V^* a V) \pi(x) \Omega \rangle,$$

where $\mathbb{E}(V^* a V)$ is interpreted as an operator-valued integral, so that

$$\Phi(a) = \int_E V^*(e) a V(e) \mu(de) = \mathbb{E}(V^* a V),$$

for all $a \in \mathfrak{A}$.

Once established the representation for completely positive maps, the next result giving the representation of completely bounded maps is quite natural.

**Theorem 3.3.5.** Let $\mathfrak{A}$ be a $C^*$-algebra with unit, and let $\Phi : \mathfrak{A} \to B(\mathfrak{h})$ be a completely bounded map. Then there exists a representation $(\pi, \mathfrak{h})$ of $\mathfrak{A}$ and bounded operators $V_i : \mathfrak{h} \to \mathfrak{k}$, $i = 1, 2$, with $\| \Phi \|_{cb} = \| V_1 \| \| V_2 \|$ such that

$$\Phi(a) = V_1^* \pi(a) V_2,$$

for all $a \in \mathfrak{A}$. If $\| \Phi \|_{cb} = 1$, then $V_1$ and $V_2$ may be taken to be isometries.
3.3. DILATIONS OF CP AND CB MAPS

Proof. We may assume \( \|\Phi\|_{cb} = 1 \) which implies that \( \Phi \) is completely contractive. We first consider a general procedure to obtain CP maps from CB maps. Introduce the operator system
\[
S = \left\{ \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} : \lambda, \mu \in \mathbb{C}, a, b \in \mathfrak{A} \right\},
\]
and define \( \Phi : S \to \mathcal{B}(\mathfrak{h} \oplus \mathfrak{h}) \) by
\[
\Phi \begin{pmatrix} \lambda 1 & a \\ b^* & \mu 1 \end{pmatrix} = \begin{pmatrix} \lambda 1 & \Phi(a) \\ \Phi(b)^* & \mu 1 \end{pmatrix}.
\]

Since \( \Phi \) is completely contractive, a direct computation using 3.1.7 shows that \( \Phi \) is completely positive and unital. Then, by Arveson Extension Theorem 3.1.5, the CP map \( \Phi \) can be extended to the whole algebra \( \mathcal{M}_2(\mathfrak{A}) \). Let \((\pi, \mathcal{V}, \mathfrak{k})\) be a minimal Stinespring representation for \( \Phi \). Since \( \Phi \) is unital, \( \mathcal{V} \) may be taken to be an isometry and \( \pi \) unital. \( \mathcal{M}_2(\mathfrak{A}) \) contains a copy of the algebra of \( 2 \times 2 \) complex matrices, and we may decompose \( \mathfrak{k} = \mathfrak{k} \oplus \mathfrak{k} \) to have \( \pi : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{B}(\mathfrak{k} \oplus \mathfrak{k}) \) of the form
\[
\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix},
\]
where \( \pi : \mathfrak{A} \to \mathcal{B}(\mathfrak{t}) \) is a unital, \( \star \)-homomorphism.

As a result, \( \mathcal{V} : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{t} \oplus \mathfrak{t} \) is an isometry and
\[
\begin{pmatrix} a & \Phi(b) \\ \Phi(c)^* & d \end{pmatrix} = \mathcal{V}^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} \mathcal{V}.
\]
The isometric property of \( \mathcal{V} \) implies that there exists a linear map \( V_1 : \mathfrak{h} \to \mathfrak{t} \), which is also an isometry and such that
\[
\mathcal{V} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} V_1 u \\ 0 \end{pmatrix}.
\]
One proves similarly the existence of \( V_2 \) such that
\[
\mathcal{V} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ V_2 v \end{pmatrix}.
\]
Thus, we finally obtain
\[
\begin{pmatrix} a & \Phi(b) \\ \Phi(c)^* & d \end{pmatrix} = \mathcal{V}^* \begin{pmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{pmatrix} \mathcal{V} = \begin{pmatrix} V_1^* \pi(a) V_1 & V_1^* \pi(b) V_2 \\ V_2^* \pi(c) V_1 & d \end{pmatrix}.
\]
\( \square \)

Completely positive maps satisfy a stronger version of Schwartz-type inequalities than the one proved before for 2-positive maps.
Theorem 3.3.6 (Schwartz-type inequalities). Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be two \( C^* \)-algebras with unit, \( \mathfrak{B} \subseteq \mathfrak{B}(\mathfrak{h}) \) where \( \mathfrak{h} \) is a separable complex Hilbert space, and let \( \Phi : \mathfrak{A} \rightarrow \mathfrak{B} \) be a linear completely positive map such that \( \Phi(1) = 1 \). Then, for all \( a_1, \ldots, a_n \in \mathfrak{A} \), \( u_1, \ldots, u_n \in \mathfrak{h} \)

\[
\sum_{i,j} \langle u_i, [\Phi(a_i^* a_j) - \Phi(a_i)^* \Phi(a_j)] u_j \rangle \geq 0. \tag{3.3.8}
\]

In particular, for all \( a \in \mathfrak{A} \):

\[
\Phi(a^* a) \geq \Phi(a)^* \Phi(a). \tag{3.3.9}
\]

Moreover, given any positive linear map \( \varphi : \mathfrak{A} \rightarrow \mathfrak{B} \) such that \( \varphi(1) = 1 \) and given any normal element \( A \in \mathfrak{A} \) it holds

\[
\varphi(A^* A) \geq \varphi(A)^* \varphi(A). \tag{3.3.10}
\]

Proof. Consider a Stinespring representation \((\pi, V)\) for the map \( \Phi \). Since \( \Phi(1) = 1 \), \( V \) is an isometry, so that \( V^* V = 1 \). Take any collection \( a_1, \ldots, a_n \in \mathfrak{A} \), \( u_1, \ldots, u_n \in \mathfrak{h} \):

\[
\sum_{i,j} \langle u_i, \Phi(a_i^* a_j) u_j \rangle = \sum_i \pi(a_i) V u_i, \sum_j \pi(a_j) V u_j \rangle
\]

\[
= \left\| \sum_i \pi(a_i) V u_i \right\|^2 
\]

\[
\geq \left\| V^* \sum_i \pi(a_i) V u_i \right\|^2 
\]

\[
= \langle \sum_i \Phi(a_i) u_i, \sum_j \Phi(a_j) u_j \rangle 
\]

\[
= \sum_{i,j} \langle u_i, \Phi(a_i)^* \Phi(a_j) u_j \rangle.
\]

The second part is an obvious consequence of the first.

For the third part, to prove the inequality in \( A \) with \( \varphi \) positive only, it is worth noticing that we can reduce \( \mathfrak{A} \) to be the abelian algebra generated by \( A \). Over that algebra, positive linear maps are completely positive and the second part of the theorem applies. \( \square \)
Chapter 4

Quantum Markov Semigroups and Flows

As we have recalled in the first lecture, an homogeneous classical Markov semigroup is characterized by a family \((P_t)_{t \geq 0}\) of Markovian transition kernels defined on a measurable space \((E, \mathcal{E})\) which satisfies Chapman-Kolmogorov equations (or the semigroup property for the composition of kernels). Given a \(\sigma\)-finite measure \(\mu\) on \((E, \mathcal{E})\), \(\mathfrak{A} = L^\infty(E, \mathcal{E}, \mu)\) represents the von Neumnan algebra of multiplication operators acting on the Hilbert space \(L^2(E, \mathcal{E}, \mu)\). In this case, the predual algebra is \(\mathfrak{A}^* = L^1(E, \mathcal{E}, \mu)\).

Moreover, \((P_t)_{t \geq 0}\) is a semigroup of completely positive maps acting on the von Neumann algebra \(\mathfrak{A}\). Additionally, this semigroup satisfies the following properties:

- It preserves the unit: \(P_t 1 = 1\), for all \(t \geq 0\).
- \(P_0 = I\), the identity mapping.
- Each \(P_t\) is \(\sigma\)-weak continuous, that is, for any increasing net \(f_\alpha\) of positive elements with upper envelope \(f\) in \(\mathfrak{A}\),
  \[
  \int_E P_t f(x) g(x) \mu(dx) = \lim_{\alpha} \int_E P_t f_\alpha(x) g(x) \mu(dx),
  \]
  for all \(g \in L^1(E, \mathcal{E}, \mu)\). Indeed, by the Monotone Convergence Theorem first, \(P_t f_\alpha(x) \uparrow P_t f(x)\), for all \(x \in E\); finally, to conclude, it is enough to apply the Dominated Convergence Theorem to \(P_t f_\alpha(x) g(x)\).

All the above properties are crucial to face the extension of Markovian concepts to a non-commutative framework. Moreover, it is well-known that the addition of suitable topological hypotheses on the space \((E, \mathcal{E})\), allows to construct a Markov process associated to a given semigroup. One can take, for instance, \(E\) to be a locally compact space with countable basis and \(\mathcal{E}\) its Borel \(\sigma\)-field. This leads to a Markovian system

\[
(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E}, (X_t)_{t \geq 0}, E, \mathcal{E})
\]
The semigroup and the process are then related by the equation
\[ P_t f(x) = E_x(f(X_t)), \]
for all \( f \in \mathfrak{A} \), \( t \geq 0 \). Moreover, we choose an arbitrary initial probability \( \nu \) on \((E, \mathcal{E})\), and denote
\[ \mathbb{P}_\nu = \int \mathbb{P}_x \nu(dx). \]
Now consider the von Neumann algebra \( \mathfrak{B} = L^\infty(\Omega, \mathcal{F}, \mathbb{P}_\nu) \). The Markov flow is defined as a \(*\)-homomorphism \( j_t : \mathfrak{A} \to \mathfrak{B} \) given by
\[ j_t(f) = f(X_t), \]
for all \( f \in \mathfrak{A} \), \( t \geq 0 \).

Inspired by these ideas we now turn into the non-commutative framework. We start by defining a Quantum Dynamical Semigroup.

### 4.1 Semigroups

Introduced by physicists during the seventies, Quantum Dynamical Semigroups (QDS) are aimed at providing a suitable mathematical framework for studying the evolution of open systems. Typically, an open quantum system involves a dissipative effect modeled through the mutual interaction of different subsystems. One commonly distinguishes between at least the “free system” and the “reservoir”.

In general a QDS can be defined over an arbitrary von Neumann algebra, as follows:

**Definition 4.1.1.** A Quantum Dynamical Semigroup (QDS) (respectively a Quantum Markov Semigroup, QMS) of a von Neumann algebra \( \mathfrak{A} \) is a weakly\(^*\)-continuous one-parameter semigroup \((T_t)_{t \geq 0}\) of completely positive linear normal maps of \( \mathfrak{A} \) into itself such that \( T_t(1) \leq 1 \) (respectively, \( T_t(1) = 1 \)). In addition, it is assumed that \( T_0 \) coincides with the identity map.

The class of semigroups defined over the von Neumann algebra \( \mathfrak{A} = B(\mathcal{H}) \) of all bounded operators over a given complex separable Hilbert space \( \mathcal{H} \), is better known. In particular, several results on the form of the infinitesimal generator of these QDS are available (see eg. [36], [13], [32]). We denote \( \mathcal{L} \) the infinitesimal generator of the semigroup \( \mathcal{T} \), whose domain is given by the set \( D(\mathcal{L}) \) of all \( X \in B(\mathcal{H}) \) for which the \( w^*\)-limit of \( t^{-1}(T_t(X) - X) \) exists when \( t \to 0 \), and we define \( \mathcal{L}(X) \) such a limit.

To have a view on the form of the generator, we consider a particular case of QDS.

**Definition 4.1.2.** A quantum dynamical semigroup \( \mathcal{T} \) is called uniformly continuous if
\[ \lim_{t \to 0} \| T_t - T_0 \| = 0. \]

### 4.2 Representation of the generator

From the general theory of semigroups it follows that a QDS is uniformly continuous if and only if its generator \( \mathcal{L} \) is a bounded operator. Within this framework the canonical form of a generator has been obtained first by Gorini, Kossakowski and Sudarshan in the finite dimensional case, extended later by Lindblad to a general Hilbert space in [36], a celebrated result which we recall below in the version of Parthasarathy ([43], Theorem 30.16).

We start by a modification of complete positivity.
4.2. REPRESENTATION OF THE GENERATOR

Definition 4.2.1. Let $\mathcal{A}$ denote a $C^*$-subalgebra of $\mathcal{B}(\mathfrak{h})$ which contains a unit. A bounded linear map $L(\cdot)$ on $\mathcal{A}$ is conditionally completely positive if for any collection $a_1, \ldots, a_n \in \mathcal{A}$ and $u_1, \ldots, u_n \in \mathfrak{h}$ such that $\sum_i a_i u_i = 0$, it holds that

$$\sum_{i,j} \langle u_i, L(a_i^* a_j) u_j \rangle \geq 0.$$

Theorem 4.2.2 (Christensen and Evans). A bounded linear map $L(\cdot)$ on the $C^*$-algebra given before such that $L(a^*) = L(a)^*$, for any $a \in \mathcal{A}$ is conditionally completely positive if and only if there exists a completely positive map $\Phi$ into its weak closure $\overline{\mathcal{A}}$ and an element $G \in \overline{\mathcal{A}}$ such that

$$L(a) = G^* a + \Phi(a) + aG,$$

for all $a \in \mathcal{A}$. Moreover the operator $G$ satisfies the inequality $G + G^* \leq L(1)$.

Proof. We restrict the proof to the case $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ for simplicity. The interested reader is referred to the original paper [11] where this result is proved for a general $C^*$-algebra.

We first take $L(\cdot)$ given by (4.2.1) and prove conditional complete positivity. Take $a_1, \ldots, a_n \in \mathcal{B}(\mathfrak{h})$, $u_1, \ldots, u_n \in \mathfrak{h}$ such that $\sum_i a_i u_i = 0$. Then

$$\sum_{i,j} \langle u_i, L(a_i^* a_j) u_j \rangle = \sum_{i,j} \langle a_i Gu_i, a_j u_j \rangle + \sum_{i,j} \langle u_i, \Phi(a_i^* a_j) u_j \rangle + \sum_{i,j} \langle a_i u_i, a_j Gu_j \rangle = \langle \sum_i a_i Gu_i, \sum_j a_j u_j \rangle + \sum_{i,j} \langle u_i, \Phi(a_i^* a_j) u_j \rangle + \langle \sum_i a_i u_i, \sum_j a_j Gu_j \rangle = \sum_{i,j} \langle u_i, \Phi(a_i^* a_j) u_j \rangle \geq 0.$$

To prove the converse, fix a unit vector $e \in \mathfrak{h}$ and define

$$G^* u = L(|u\rangle\langle e|) e - \frac{1}{2} \langle e, L(|e\rangle\langle e|) e \rangle u,$$

for all $u \in \mathfrak{h}$. Given $a_1, \ldots, a_n \in \mathcal{B}(\mathfrak{h})$, $u_1, \ldots, u_n \in \mathfrak{h}$, let

$$u_{n+1} = e, \quad v = - \sum_{j=1}^n a_j u_j, \quad a_{n+1} = |v\rangle\langle e|.$$

(4.2.2) (4.2.3) (4.2.4)
Then $\sum_{j=1}^{n+1} a_j u_j = 0$. Since $\mathcal{L}(\cdot)$ is conditionally completely positive,

$$0 \leq \sum_{i,j=1}^{n} \langle u_i, \mathcal{L}(a_i^* a_j) u_j \rangle$$

$$+ \sum_{i=1}^{n} \langle u_i, \mathcal{L}(a_i^* v) \langle e \rangle e \rangle$$

$$+ \sum_{j=1}^{n} \langle e, \mathcal{L}(\langle e \rangle \langle e \rangle e \rangle) u_j \rangle$$

$$+ \langle e, \mathcal{L}(\langle e \rangle \langle e \rangle e \rangle) \|v\|^2.$$ 

Using the definition of $G^*$, the sum of the last three terms becomes

$$\sum_{i,j=1}^{n} \langle u_i, G^*(a_i^* v) u_j \rangle + \sum_{j=1}^{n} \langle a_i^* a_j u_j \rangle = -\sum_{i,j} \langle u_i, G^* a_i^* a_j u_j \rangle - \sum_{i,j} (u_i, a_i^* a_j G u_j).$$

If we define $\Phi(a) = \mathcal{L}(a) - G^* a - a G$, the inequality we obtained here before can be written

$$\sum_{i,j=1}^{n} \langle u_i, \Phi(a_i^* a_j) u_j \rangle \geq 0,$$

which means that $\Phi$ is completely positive and the Theorem is proved.

Assume $\mathcal{T}$ to be a norm continuous quantum Markov semigroup on $\mathcal{B}(\mathcal{h})$. By the Schwartz inequalities, for any $a_1, \ldots, a_n \in \mathcal{B}(\mathcal{h}), u_1, \ldots, u_n \in \mathcal{h}$, and any $t \geq 0$:

$$\sum_{i,j=1}^{n} \langle u_i, (\mathcal{T}_t(a_i^* a_j) - \mathcal{T}_t(a_i^*) \mathcal{T}_t(a_j)) u_j \rangle \geq 0$$

The norm continuity of $\mathcal{T}$ implies that $\mathcal{L}(\cdot)$ is defined as a bounded operator on the whole algebra $\mathcal{B}(\mathcal{h})$, so that the above inequality implies

$$\sum_{i,j=1}^{n} \langle u_i, (\mathcal{L}(a_i^* a_j) - \mathcal{L}(a_i^*) a_j - a_i^* \mathcal{L}(a_i)) u_j \rangle \geq 0,$$

from which, if $\sum_i a_i u_i = 0$, it follows easily

$$\sum_{i,j=1}^{n} \langle u_i, \mathcal{L}(a_i^* a_j) u_j \rangle \geq 0.$$

So that $\mathcal{L}(\cdot)$ is conditionally completely positive. As a result, the following characterization follows.
4.2. REPRESENTATION OF THE GENERATOR

**Theorem 4.2.3.** Given a norm continuous quantum dynamical semigroup \( T \) on \( \mathcal{B}(\mathfrak{h}) \), there exists an operator \( G \) and a completely positive map \( \Phi \) such that its generator is represented as

\[
\mathcal{L}(x) = G^* x + \Phi(x) + xG, \quad (x \in \mathcal{B}(\mathfrak{h})).
\] (4.2.5)

Since \( \mathcal{B}(\mathfrak{h}) \) is a von Neumann algebra, the representation before can be improved using Kraus Theorem to represent the completely positive map \( \Phi \).

**Theorem 4.2.4 (Lindblad).** Let be given a uniformly continuous quantum dynamical semigroup on the algebra \( \mathcal{B}(\mathfrak{h}) \) of a complex separable Hilbert space \( \mathfrak{h} \). Let \( \rho \) be any state in \( \mathfrak{h} \).

Then there exists a bounded self-adjoint operator \( H \) and a sequence \((L_k)_{k \in \mathbb{N}}\) of elements in \( \mathcal{B}(\mathfrak{h}) \) which satisfy

1. \( \text{tr} \rho L_k = 0 \) for each \( k \);
2. \( \sum_k L_k^* L_k \) is a strongly convergent sum;
3. If \( \sum_k |c_k|^2 < \infty \) and \( c_0 + \sum_k c_k L_k = 0 \) for scalars \( c_k \), then \( c_k = 0 \) for all \( k \);
4. The generator \( \mathcal{L} \) of the semigroup admit the representation

\[
\mathcal{L}(X) = i[H,X] - \frac{1}{2} \sum_k (L_k^* L_k X - 2L_k^* X L_k + X L_k^* L_k),
\]

for all \( X \in \mathcal{B}(\mathfrak{h}) \).

This result has been extended by Davies (see [13]) to a class of QDS with unbounded generators.

Generators of QDS commonly appear in Physics articles in its *predual* form. That is, given the von Neumann algebra \( \mathfrak{A} = \mathcal{B}(\mathfrak{h}) \) its *predual space* consists of \( \mathfrak{A}_* = \mathcal{T}^1(\mathfrak{h}) \) the Banach space of trace-class operators. A quantum dynamical semigroup \( T \) induces a *predual* semigroup \( T_* \) on \( \mathfrak{A}_* \) given through the relation

\[
\text{tr} (T_*(Y)X) = \text{tr} (Y T_1(X)),
\]

for any \( Y \in \mathfrak{A}_* \), \( X \in \mathfrak{A} \).

The generator of the preduel semigroup is denoted \( \mathcal{L}_* \). What is usually called a *master equation* in Open Quantum Systems, is referred to the relation between the preduel semigroup and its generator, written in the form

\[
\frac{d}{dt} \rho_t = \mathcal{L}_*(\rho_t),
\]

where \( \rho_t = T_*(\rho) \), for any \( t \geq 0 \), \( \rho \) being a state, that is, an element \( \rho \in \mathfrak{A}_* \) with unitary trace.

**Example 8.** Coming back to the basic closed quantum dynamics, we consider the space \( \mathfrak{h} = \mathbb{C}^2 \) and the basis \( e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Call

\[
H = |e_1\rangle\langle e_1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

As we mentioned at the beginning of section 2, the (closed) quantum dynamics associated to the Hamiltonian \( H \) is defined through a group of unitary operators \( U_t : \mathfrak{h} \to \mathfrak{h} \) given by \( U_t = \exp(-itH) \),
(\(t \in \mathbb{R}\)), that is,

\[
U_t = \begin{pmatrix} 1 & 0 \\ 0 & e^{-it} \end{pmatrix}.
\]

Equivalently, the operator \(H\) defines an automorphism group \(\alpha_t\) on the algebra \(B(\mathfrak{h})\) of all linear (bounded) operators on \(\mathfrak{h}\) which is isomorphic with \(M_2(\mathbb{C})\) the algebra of two by two complex matrices:

\[
\alpha_t(x) = U_t^* x U_t, \quad (t \in \mathbb{R}).
\]

This is the so called Heisenberg picture of the dynamics, while its predual version \(\alpha_t^*(\rho) = U_t \rho U_t^*\) defined on unit trace operators \(\rho\), bears the name of Schrödinger. Consider a positive operator with unit trace \(\rho = \begin{pmatrix} p e^{i\theta} & q \\ re^{-i\theta} & q \end{pmatrix}\), where \(r > 0, \theta \in [0, 2\pi], 0 \leq r \leq 1/2\) and \((p - q)^2 \leq 1 - 4r^2\), \(p, q > 0, p + q = 1\).

The evolution of \(\rho\) at time \(t\) is then given by

\[
\alpha_{st}(\rho) = \begin{pmatrix} p e^{-i(t+\theta)} & q e^{i(t+\theta)} \\ re^{-i(t+\theta)} & q \end{pmatrix}.
\]

The generator of the group \(\alpha_s\) (respectively \(\alpha\)) is \(\delta_s(\rho) = i[H, \rho]\), (resp. \(\delta(x) = i[H, x]\) for all endomorphism \(x\)).

Any diagonal state operator \(\rho = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\) is invariant, that is \(\delta_s(\rho) = 0\).

Now we open the system, that is, interaction with the environment is allowed, so that it is embedded as a “small subsystem” in a bigger structure that we call the total system. That is, the environment is supposed to be a “big” subsystem which is not observed but supposed in equilibrium, represented by an equilibrium state \(\rho_\beta\). We face a similar situation that the one described in section 1, where we started from a classical closed dynamics, and added perturbations originated in environment interactions.

The interaction of our small subsystem with the environment introduces a perturbation in both of them. We will assume that the environment returns to equilibrium much faster than the time scale of the small system evolution.

Given a state \(\rho\) on the initial space \(\mathfrak{h}\), the reduced dynamics is obtained by performing a limit procedure on the time scale evolution of the environment and a partial trace of the total dynamics. This is the so called Markov approximation of the open system dynamics.

To give a rough picture of the approximation, under suitable hypotheses one obtains a limit dynamics \(\bar{U}_t\) defined on the space \(\mathfrak{h} \otimes \mathfrak{h}_R\), where \(\mathfrak{h}_R\) is the Hilbert space associated to the reservoir. Then the evolution of the state \(\rho\) is given by the following partial trace on the reservoir variables:

\[
\mathcal{T}_{st}(\rho) = \text{tr}_R \left( \bar{U}_t \rho \otimes \rho_\beta \bar{U}_t^* \right).
\]

The dual version of the above expression gives the evolution \(\mathcal{T}_t(x)\) of any observable \(x\):

\[
\text{tr}(\mathcal{T}_{st}(\rho)x) = \text{tr}(\rho \mathcal{T}_t(x)), \quad (t \geq 0).
\]

It turns out that \(\mathcal{T}\) above defines a semigroup structure on \(M_2(\mathbb{C})\). This semigroup (resp. its dual \(\mathcal{T}\)) has a generator that can be written in the form of Theorem 4.2.4.

\[
\mathcal{L}(x) = i[H, x] + \mathcal{D}(x), \quad (4.2.7)
\]
4.2. REPRESENTATION OF THE GENERATOR

\( \mathcal{L}(\rho) = i[\rho, H] + \mathcal{D}(\rho) \), where \( \mathcal{D} \) (resp. \( \mathcal{D}_* \)) represents the dissipation due to the interaction of the system with the reservoir. So that, for instance, assume that the dissipation is written in the so called \textit{Gorini-Kossakowski-Sudarshan-Lindblad form} as follows:

\[
\mathcal{D}(x) = -\frac{1}{2}(\sigma_+ \sigma_- x - 2\sigma_+ x \sigma_- + x \sigma_+ \sigma_-),
\]

where

\[
\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
\sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Notice that \( \sigma_+ \sigma_- = |e_1 \rangle \langle e_1| \). The generator is then

\[
\mathcal{L}(x) = i|e_1 \rangle \langle e_1| x - \frac{1}{2}(|e_1 \rangle \langle e_1| x - 2\sigma_+ x \sigma_- + x |e_1 \rangle \langle e_1|). \tag{4.2.9}
\]

And that of the predual semigroup,

\[
\mathcal{L}_*(\rho) = i|e_1 \rangle \langle e_1| \rho - \frac{1}{2}(|e_1 \rangle \langle e_1| \rho - 2\sigma_- \rho \sigma_+ + \rho |e_1 \rangle \langle e_1|). \tag{4.2.10}
\]

Now, a diagonal state \( \rho = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \) is no more invariant, indeed \( \mathcal{L}_*(\rho) = \begin{pmatrix} q & 0 \\ 0 & -q \end{pmatrix} \). So that \( \rho_\infty \) is invariant if and only if it is of the form

\[
\rho_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |e_0 \rangle \langle e_0|.
\]

A state \( \rho \) is \textit{faithful} if for all positive operator \( a \) such that \( \text{tr}(\rho a) = 0 \) implies \( a = 0 \).

It is clear that \( \rho_\infty \) before is not faithful, since \( \text{tr}(\rho_\infty |e_1 \rangle \langle e_1|) = 0 \) whereas \( |e_1 \rangle \langle e_1| \neq 0 \).

Remark 4.2.5. The above examples provide a partial view on quantum dynamical semigroups, in all these cases the generators are bounded which is not satisfactory from the point of view of physical applications. The theory has been extended to unbounded (form-like) generators, obtained via phenomenological assumption or, via first principles by means of limit procedures (weak coupling or low density limits, included in the concept of stochastic limit in [1]).

As a counterpart to classical noises like Brownian Motion or Poisson processes, there exists \textit{quantum noises} (creation, annihilation, number operators) that are more suitable for the description of open quantum systems. Quantum noises appear naturally within the framework of Fock spaces. Numerous authors (see for instance [38]) have stressed the main advantage of a (boson) Fock space: that structure supports both, the Canonical Commutation Relations (CCR) and a theory of stochastic integration with respect to quantum noises providing a non commutative version of Itô’s algebra for differentials.

In the following example we show a more sophisticated generator of a Quantum Markov Semigroup. This will be given as form, since the coefficients are unbounded operators, we skip here the proof that there exists a semigroup with that generator. That property follows as a particular case of a more general result proved by Davies (see [13], [12]) and [20] for a more detailed explanation).
Example 9. Consider \( \mathfrak{h} = \ell^2(\mathbb{N}) \) with its canonical orthonormal basis \((e_n)_{n \in \mathbb{N}}; \mathfrak{M} = \mathcal{B}(\mathfrak{h}), \varphi(x) = \text{tr}(\rho^x)\).

We remind the customary notations for annihilation \((a)\), creation \((a^\dagger)\) and number \((N)\) operators. Consider a two-level atom, and denote \( A \) the energy decay rate; \( \nu \), the number of thermal excitations; \( \omega \), the natural (circular) frequency. The form-generator of the semigroup is given by the (formal) expression

\[
L(x) = i[\omega N, x] - \frac{1}{2} A(\nu + 1) (a^\dagger ax - 2a^\dagger xa + xa^\dagger a) - \frac{1}{2} A\nu (aa^\dagger x - 2axa^\dagger + xaa^\dagger),
\]

for \( x \) in a dense subset of \( \mathfrak{M} \), which is the common domain of \( a \) and \( a^\dagger \).

Denote \( T = (T_t)_{t \geq 0} \) the quantum Markov semigroup generated by the above form-generator. This semigroup leaves invariant the algebra generated by the number operator. Indeed, the spectrum of this operator is \( \mathbb{N} \), the elements \( e_n \) are the eigenvectors of \( N \) and for any bounded function \( f : \mathbb{N} \rightarrow \mathbb{C} \) a straightforward computation yields

\[
\mathcal{L}(f(N))(v, |e_n\rangle \langle e_n| u) = \mathcal{L}(f(N))(|e_n\rangle \langle e_n| v, u) = Lf(n)\langle v, e_n \rangle \langle e_n, u \rangle,
\]

where,

\[
Lf(n) = \lambda_n(f(n + 1) - f(n)) + \mu_n(f(n - 1) - f(n)),
\]

and

\[
\lambda_n = A\nu(n + 1), \quad \mu_n = A(\nu + 1)n, \quad (n \in \mathbb{N}).
\]

As it is easily seen, the expression (4.2.12) corresponds to the generator of a classical birth-and-death Markov semigroup, with birth rate \( \lambda_n \) and death rate \( \mu_n \). So, the classical dynamics is included as a particular case of the quantum one (when the semigroup is reduced to the subalgebra of the number operator).

The algebra generated by \( a, a^\dagger \) and \( 1 \) is topologically irreducible, that means that the commutant is \( \mathbb{C} \). The birth and death semigroup has a unique faithful stationary probability measure \( p \) since \( \lambda_n < \mu_n \), for all \( n \) and it is recurrent [24]. Moreover, the explicit computation of the stationary probability measure provides the following expression for the unique faithful stationary state:

\[
\rho_\infty = p(N) = \frac{1}{\nu + 1} \left( \frac{\nu}{\nu + 1} \right)^N.
\]

Remark 4.2.6. As in the classical case, a semigroup is obtained as projection of a flow, which satisfies a stochastic differential equation. Hudson and Parthasarathy were among the first to study stochastic differential equations driven by quantum noises (see [43]), thus inaugurating the field of quantum stochastic differential equations, while others investigated quantum equations driven by classical noises (see for instance [32]). Those foundational results attracted a part of the mathematical community by the end of the last century. So, the study of linear quantum stochastic differential equations with unbounded coefficients have been done by several authors. Namely, Fagnola in [15] established a useful criterion on the existence and uniqueness of solutions to equations of the form

\[
dV(t) = V(t) \sum_{\ell, m} L_{\ell}^m d\Lambda_{m}(t),
\]

where the processes \( \Lambda_{m}^\ell \) are quantum noises.
4.3 Conclusions and outlook. An invitation to further reading

Open System Theory provides a rich challenge to mathematicians. As we mentioned, one starts by defining a main system -supporting our observables- and write down the main dynamics. This main system is continuously interacting with the environment -which is not fully observed- and so, considered as a noise. As a result, all dynamics proposed for the main system is perturbed by the environment and one needs to include a suitable description of this noise interaction in the mathematical model. Here, we briefly explained an algebraic setting allowing to characterize the open system dynamics. The interested reader is referred to the three volumes [5] of a summer school held in Grenoble, which contains a suitable starting point of the theory. A Markov semigroup is the concept which is most commonly used to represent an open system dynamics, classical or quantum.

There is currently a significant progress in the Theory of Quantum Markov Semigroups, particularly in the analysis of large-time behavior of open quantum dynamics [21]. It is impossible to dress here an exhaustive list of contributions and subjects covered in this field at present. We restrict our panorama referring a number of problems which have been investigated by our team in recent years:

- Examples in Quantum Optics: [17], [19];
- Criteria on the existence of stationary states based on the analysis of the generator of the semigroup: [22],
- Existence of faithful stationary states: [23],
- The convergence towards the stationary state (ergodicity): [27], [18], [25],
- Recurrence and transience: [24],
- The problem of quantum decoherence: [6], [48], [49], [14], [2],
- The entropy production, time reversal and detailed balance: [26],
- Classical reductions and classical dilations: [50],
- Non-linear stochastic Schrödinger equations: [39].
Bibliography


